A Study of Vertex Operator Constructions for Some Infinite Dimensional Lie Algebras

A Thesis Submitted to the College of Graduate Studies and Research

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

In the Department of Mathematics and Statistics

of the University of Saskatchewan

By

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Saskatoon. Canada

1998

0-612-27433-0

UNIVERSITY OF SASKATCHEWAN
COLLEGE OF GRADUATE STUDIES AND RESEARCH

Saskatoon

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Dedicated to

my wife Zhiwei Chen with love

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ABSTRACT

In Chapter one of the thesis we construct a module for the toroidal Lie algebra and the *extended toroidal Lie algebra* of type A_1. The *Fock module* representation obtained here is faithful and completely reducible over the extended toroidal Lie algebra. We also study the level two vertex operator representation of the toroidal Lie algebra of type A_1. This generalizes the Lepowsky-Wilson study of the principal level two standard module for $A_1^{(1)}$. In Chapter two we present a complete description of the TKK algebra $\hat{\mathcal{K}}(\mathcal{T}(S))$, which allows us in Chapter three to give a faithful representation to this Lie algebra by vertex operators. In the construction of this TKK algebra by vertex operators the Clifford algebra enters the picture. The situation here is similar to, but more complicated than, that for the level 2 standard $A_1^{(1)}$-module and the level 1 standard $B_l^{(1)}$-module, where the Lie algebras of operators act on a vector space of mixed boson-fermion states. In the last chapter of the thesis we give two constructions for the toroidal Lie algebra of type B_l ($l \geq 3$) by vertex operators. The first construction is related to the folding of Dynkin diagram of $D_{l+1}^{(1)}$ and a two-cocycle necessary for the vertex operator construction. This construction also suggests a direct construction of the toroidal Lie algebra of type B_l by vertex operators. In fact, the second construction generalizes the Lepowsky-Primc construction of the level one standard module of $B_l^{(1)}$ to the toroidal case.

Acknowledgements

I would like to express my deep thanks to my supervisor Professor Stephen Berman for his constructive advice, patient guidance and constant encouragement during my study and the development of my thesis.

Also, my thanks are expressed to Professor M. Bremner, Professor J.A. Brooke, Professor A. Cason, Professor M. Khoshkam, Professor J. Martin, Professor J. Szimigeski, Professor K. Talyor and Professor E.D Tymchatin from whom I have learned a lot through their graduate classes.

I wish to thank all members of my advisory committee for their reading of this thesis. In particular, I would like to thank Professor Robert Moody for his interest in my work and for being my thesis external examiner.

Finally, My thanks go to the Department of Mathematics and Statistics, and the College of Graduate Studies and Research for financial support during my study.

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Chapter 4: Vertex Operator Representations for Toroidal Lie Algebra of Type B_l

INTRODUCTION

In this thesis we study the vertex operator representations for some infinite dimensional Lie algebras related to the extended affine Lie algebras. The thesis consists of four chapters. In Chapter 1 we give two constructions of the toroidal Lie algebras of type A_1 by twisted vertex operators. Chapter 2 is devoted to the study of the universal central extension $\hat{\mathcal{G}}(\mathcal{T})$ [see Proposition 2.3.2] of the Tits-Kantor-Koecher algebra (or TKK algebra for short) for certain Jordan algebra $\mathcal{T}$ which comes from semilattices as in [AABGP]. The structure of $\hat{\mathcal{G}}(\mathcal{T})$ will be described in terms of power series identities. This allows us to go on, in Chapter 3, to study the vertex operator representation of $\hat{\mathcal{G}}(\mathcal{T})$. Finally, in Chapter 4 we study the vertex operator representations of the toroidal Lie algebra of type B_l. Indeed, two constructions of this algebra are given in this chapter.

In the late sixties Kac [K1] and Moody [Mo] independently introduced and studied a new class of infinite-dimensional Lie algebras, which are now universally called Kac-Moody algebras. With important applications in nonlinear evolution equations, number theory, statistical mechanics, and quantum field theory, the theory of Kac-Moody algebras has become a major focus of interest in both mathematics and physics. In the past twenty years these algebras have been studied systematically both in depth and in scope. Since then, some generalized notions of Kac-Moody algebras have arisen from different contexts. To give a brief description, we first recall the definitions of Kac-Moody algebras and some of their generalizations.

An integral $n \times n$ matrix $A = (a_{ij})$ is called a generalized Cartan matrix if it satisfies the following conditions:

$$a_{ii} = 2, \text{ for } i = 1, 2, \ldots, n,$$

$$a_{ij} \leq 0, \text{ for all } i \neq j,$$

$$a_{ij} = 0 \text{ implies } a_{ji} = 0,$$

for $i, j = 1, 2, \ldots, n$. Let $\mathcal{G}'(A)$ be the complex Lie algebra with generators e_i, f_i, h_i, $i =$

$1, 2, \ldots, n$, and the following relations:

$$[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0,$$

$$[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j,$$

for $i, j = 1, 2, \ldots, n$. Then the Kac-Moody algebra $\mathcal{G}(A)$, associated with the generalized Cartan matrix A, is the quotient of $\mathcal{G}'(A)$ by the largest ideal which intersects the span of $h_1, h_2, \ldots, h_n$ trivially. In particular, if the matrix A is indecomposable and has nullity one [cf [K2],[MP]], then $\mathcal{G}(A)$ forms an important class of Kac-Moody algebras, which are called the affine Lie algebras. From [K2][MP], one can see a concrete construction of all affine Lie algebras. In fact, for a simple finite-dimensional Lie algebras $\mathcal{G}$, the corresponding affine Lie algebra $\mathcal{G}^{(1)}$ is isomorphic to the universal central extension of the loop algebra

$$\mathcal{G} \otimes_{\mathbf{C}} \mathcal{A},$$

where $\mathcal{A} = \mathbf{C}[t, t^{-1}]$.

As a generalization of Kac-Moody algebras, one extends the definition of a generalized Cartan matrix to be an integral matrix $A = (a_{ij})$ such that

$a_{ii} = 2$, for $i = 1, 2, \ldots, n$,

$a_{ij} < 0$(resp. > 0) implies $a_{ji} < 0$(resp. > 0), for $i, j = 1, 2, \ldots, n$,

then the Lie algebra $\mathcal{G}(A)$ attached to this matrix A with some generators and relations [cf. [Sl1][BM][BZ][Ga1]] form an interesting class of infinite-dimensional Lie algebras. These Lie algebras, now known as the generalized intersection matrix algebras (or GIM algebras for short), were first developed by Slodowy [Sl1] and motivated by the theory of singularities.

On the other hand, if one extends the definition of a generalized Cartan matrix to be a real matrix $A = (a_{ij})_{i,j \in I}$, here I is a (finite or) countable index set, satisfying the following conditions:

A is symmetrizable,

$a_{ij} \leq 0$ if $i \neq j$.

if $a_{ii} > 0$, then $\frac{2a_{ij}}{a_{ii}} \in \mathbf{Z}$,

for $i, j \in I$, then the Lie algebra $\mathcal{G}(A)$ associated to the matrix A with some generators and relations [cf. [Bo1][JLW]] also forms a generalized Kac-Moody algebra, which is sometimes called a Borcherds algebra. This class of generalized Kac-Moody algebras is of great interest because it contains a spacial infinite-dimensional Lie algebra, the monster Lie algebra. Borcherds [Bo2] uses this monster Lie algebra to give a positive answer to the Conway-Norton conjecture [CN].

Another natural generalization of Kac-Moody algebras is the toroidal Lie algebras, which are defined and studied by Moody, Rao and Yokonuma [MRY] [EM](see also [Fr1][MS][Sl2][Wa]). A toroidal Lie algebra is a perfect central extension of the Lie algebra $\mathcal{G} \otimes A_\nu$ where $\mathcal{G}$ is a finite dimensional simple Lie algebra over $\mathbf{C}$, and A_ν is the ring of Laurent polynomials in commuting variables $t_0, t_1, \ldots, t_\nu$, $\nu \geq 0$ over $\mathbf{C}$. When $\nu = 0$ they are precisely the untwisted affine Kac-Moody algebras which have been studied intensively both from the physical and mathematical points of view.

In 1990 Hoegh-Krohn and Torresani[H-KT] introduced a new class of Lie algebras under the name quasi-simple Lie algebras in an attempt to generalize the affine Kac-Moody algebras. These Lie algebras are now called the extended affine Lie algebras (or EALA's for short). The EALA's are indeed natural generalization of the finite-dimensional simple Lie algebras, the affine Lie algebras as well as the toroidal Lie algebras. The first classification results on EALA's were proved in [BGK] and [BGKN] (see also [Ga3]). Recently, a systematic study of the classification of the EALA's has been developed in [AABGP] (see also [AG] and [Sa]).

In the study of the affine Kac-Moody algebras, one important direction of research, after the classification, is the construction of standard modules by vertex operator representations, which give an explicit contruction of the standard modules. Applications of this allows construction of soliton solutions for some nonlinear evolution equations such as the nonlinear Korteweg-de Vries equations and the nonlinear

Schrodinger equations (see [DKM][K2]).

Vertex operators were discovered by physicists in the study of string theory. Lepowsky and Wilson [LW1] were the first to discover the construction of the affine Lie algebra $A_1^{(1)}$ by using vertex operators. Soon after [LW1] the construction was generalized by Kac, Kazhdan, Lepowsky and Wilson [KKLW] to all simply-laced affine Lie algebra of type A, D, E, and Frenkel [Fr2] gave a new impetus to the boson-fermion correspondence. The modules constructed in [LW1][KKLW] are often called the principal level one standard modules. The same level one modules of the affine Lie algebras of type A, D, E, in the homogeneous picture, were constructed by Frankel and Kac in [FK] and also by Segal [Seg].

For the toroidal Lie algebras the homogeneous construction by vertex operators was first studied by Moody, Rao and Yokonuma [MRY][EM]. In these papers they constructed faithful vertex operator representations for the universal central extension of $\mathcal{G} \otimes A_\nu$, where $\mathcal{G}$ is a simply-laced finite dimensional simple Lie algebra over $\mathbf{C}$. This whole construction is a generalization of the Frenkel-Kac and Segal level one constructions for the affine Kac-Moody algebras. Unlike the affine case, the universal central extension of $\mathcal{G} \otimes A_\nu$, $\nu \geq 1$, has a infinite dimensional center, and the vertex operator representations [MRY][EM] are, in fact, not completely reducible over the toroidal Lie algebras. Recently, Billig [Bi1] generalized the level one construction by Kac-Kazhdan-Lepowsky-Wilson for the affine case, to obtain a principal vertex operator representation of the toroidal Lie algebras. In [Bi1] some derivations of the Laurent ploynomial ring A_ν are added to the toroidal Lie algebra to form an extended algebra. The vertex operator representation is irreducible over this extended toroidal Lie algebra. As an application, Billig [Bi2] constructs an extension of the KdV hierarchy from the vertex operator representation given in [Bi1].

Among the affine Kac-Moody algebras, $A_1^{(1)}$ is the simplest. Discoveries in the case $A_1^{(1)}$ have frequently led to fruitful new directions for the general theory. Due to this reason, in Chapter 1 we will consider an analogue of the level one and level two representations for the affine Kac-Moody algebra $A_1^{(1)}$ by twisted vertex operators. As in [EMY][FLM], we start with a finite rank even lattice Γ, equipped with a non-degenerate symmetric form, and form a Heisenberg algebra $\hat{\mathcal{H}}$. Using $\hat{\mathcal{H}}$ and an extension of the standard Fock module of $\hat{\mathcal{H}}$, we construct a module for both the toroidal Lie algebra and the extended toroidal algebra of type A_1. The Fock module representation obtained here is faithful, and is completely reducible over the extended toroidal Lie algebra, and all its irrducible constituents are indeed isomorphic to the corresponding module obtained in [Bi1]. In fact, it was this work [Bi1] which provided me with some of the necessary insight to obtain the vertex operator representations which are studied here.

We will also study the level two vertex operator representation of the toroidal Lie algebra of type A_1. This generalizes the Lepowsky-Wilson study [LW2] of the principal level two standard module for $A_1^{(1)}$. We do this from a fermionic point of view, and to the best of our knowledge this is the first time that the fermonic framework has been used for the toroidal Lie algebras. This framework also comes into Chapter 3 and Chapter 4.

In Chapter 2 we study the universal central extension of TKK algebras. The TKK algebra $\mathcal{K}(\mathcal{J})$ is obtained from a Jordan algebra $\mathcal{J}$ by using the Tits-Kantor-Koecher construction (see[J1, VIII.5]). In [AABGP], they start with a semilattice S to define a Jordan algebra $\mathcal{J} = \mathcal{J}(S)$. From $\mathcal{J}(S)$ they contruct an example of an extended affine Lie algebra with an extended affine root system of type A_1.

When one specializes this construction to the smallest possible semi-lattice in two variables, one obtains the smallest extended affine Lie algebra which is not of finite or affine type. That is, it has the smallest possible root system which is not finite or affine (see [AABGP]). For this reason we will call the extended affine Lie algebra arising from this semilattice the Baby algebra. We study this Baby algebra extensively in Chapter 2 and Chapter 3.

Note that in studying the affine theory it was the case that the algebra $A_1^{(1)}$ was studied first, and it makes good sense to follow this program here. However, from a cursory knowledge of the class of extended affine Lie algebras one might think that the toroidal algebra of type A_1 should be the first to be studied, and indeed this is one reason that they occupied our attention in Chapter 1. We feel that a good knowledge of this algebra and the Baby algebra is necessary for the development of further theory.

We will see (in Proposition 2.2.3) that $\mathcal{K}(\mathcal{J})$ is isomorphic to the following Lie algebra

$$\mathcal{G}(\mathcal{J}) := sl_2(\mathbb{C}) \otimes \mathcal{J} \dotplus \mathrm{Inder}(\mathcal{J}),$$

where $\mathrm{Inder}(\mathcal{J})$ is the set of inner derivations of the Jordan algebra $\mathcal{J}$. Thus this TKK algebra $\mathcal{K}(\mathcal{J})$ comes from the smallest finite dimensional simple Lie algebra $sl_2(\mathbb{C})$. We will indeed concerntrate on a Jordan algebra $\mathcal{J} := \mathcal{J}(S)$, where S is the 'smallest' (non-lattice) semilattice, and this will be the coordinate algebra of the Baby algebra. We close Chapter 2 by describing the structure of $\hat{\mathcal{G}}(\mathcal{J})$, the universal central extension of the TKK algebra $\mathcal{G}(\mathcal{T})$, in terms of formal power series identities in formal variables. This will allow us to go on to study the vertex operator representation of $\hat{\mathcal{G}}(\mathcal{J})$.

In Chapter 3 we construct a module for the universal central extension $\hat{\mathcal{G}}(\mathcal{J})$ of the TKK algebra $\mathcal{G}(\mathcal{J})$ and the extended TKK algebra $\tilde{\mathcal{G}}(\mathcal{J})$ (see (see (3.4.2)) by vertex operators. In fact, if K is the ideal of $\tilde{\mathcal{G}}(\mathcal{J})$ defined in Section 3.4, then the quotient $\tilde{\mathcal{G}}(\mathcal{J})/K$ is isomorphic to the extended affine Lie algebra of type A_1 given in [III.2 in AABGP]. Moreover, an interesting consequence of our results is that two vertex operator representations of the toroidal Lie algebra of type A_1 in the principal and homogeneous pictures are direct consequence of the vertex operator construction of the Lie algebra $\hat{\mathcal{G}}(\mathcal{J})$ (see Section 3.5).

Finally, in Chapter 4 we study the vertex operator representations for the toroidal Lie algebra of type B_l ($l \geq 3$). We will give two constructions of this algebra, we start with an integral lattice Q which indeed contains the affine root lattices $Q(D_{l+1}^{(1)})$

and $Q(B_l^{(1)})$ as sublattices. To define the full Fock space V [see (4.3.6)] which affords a representation of the toroidal Lie algebra of type B_l by vertex operators. we define a group algebra $\mathbb{C}[Q]$ of Q with the twisted multiplication $e^{\beta}e^{\gamma} = \epsilon(\beta,\gamma)e^{\beta+\gamma}$ for $\beta,\gamma \in Q$. where the map ϵ: $Q \times Q \to \{\kappa | \ \kappa^4 = 1\}$ satisfies the 2-cocycle condition [see Lemma 4.3.1]. It follows from the construction of this algebra that the Fock space V indeed also afford a vertex operator representation (see Theorem 4.5.2) of the Clifford algebra $\mathcal{W}$. which is spanned by the elements ω_j ($j \in 2\mathbf{Z}+1$) with the relation $\omega_i\omega_j + \omega_j\omega_i = -2\delta_{i+j,0}$. ($i,j \in 2\mathbf{Z}+1$). The construction studied in Section 4.3 and Section 4.4 suggests a direct construction of the toroidal Lie algebra of type B_l (see Theorem 4.6.1). In fact. this construction generalizes the Lepowsky-Primc construction of the level one standard module of $B_l^{(1)}$ [LP1] to the toroidal case.

As the basic set up in the study of vertex operator representations is well known in the affine Lie algebras case. we will be brief and assume the basics on notation. and delta function identities. As has become customary we will only state the notions and fundamental results which we will make use of. For details one may consult [FLM]. However. for the benefit of the reader we provide all necessary references.

Chapter 1

Principal Constructions of the Toroidal Lie Algebra of Type A_1

§1.1 Introduction

In this chapter, we will consider an analogue of the level one and level two principal representations for the affine Kac-Moody algebra $A_1^{(1)}$ by vertex operators. As in [EMY][FLM], we start with a finite rank even lattice Γ, equipped with a non-degenerate symmetric form, and form a Heisenberg algebra $\hat{\mathcal{H}}$. Using $\hat{\mathcal{H}}$ and an extension of the standard Fock module of $\hat{\mathcal{H}}$, we construct a module for both the toroidal Lie algebra and the extended toroidal algebra of type A_1. The Fock module representation obtained here is faithful, and is completely reducible over an extended toroidal Lie algebra.

We will also study the level two vertex operator representation of the toroidal Lie algebra of type A_1. This generalizes the Lepowsky-Wilson study of the principal level two standard module for $A_1^{(1)}$, see [LW2].

This chapter is organized as follows. In next section we set up some notation to be used in this chapter and then we recall the structure of the toroidal Lie algebra $\hat{\mathcal{G}}$ of type A_1, which is the universial central extension of $sl_2(\mathbb{C}) \otimes A_\nu$. The structure of $\hat{\mathcal{G}}$ will be described in terms of formal power series in commuting formal variables. In Section 1.3 we define an even lattice Γ, and from which we form a (principal) Heisenberg algebra $\hat{\mathcal{H}}$, and its standard irreducible Fock module $\mathcal{S}(\hat{\mathcal{H}}^-)$. Section 1.4 is devoted to the definition of the vertex operators. We close this section with the main results, Theorem 1.4.2 and Theorem 1.4.3, of this chapter. Section 1.5 and Section 1.6 are concerned with the proof of Theorem 1.4.2 and Theorem 1.4.3 respectively. In Section 1.7 we prove that the toroidal Lie algebra $\hat{\mathcal{G}}$ is faithfully represented by the vertex operators. The structure of the module is also studied in this section. Finally, in Section 1.8 we study the level two module of $\hat{\mathcal{G}}$.

§1.2 The Toroidal Lie Algebra $\hat{\mathcal{G}}[\theta]$ of Type A_1

We will use the notation $\mathbf{N}$ for the set of natural numbers $\{0, 1, 2, \ldots\}$, and $\mathbf{Z}, \mathbf{Z}_+, \mathbf{Z}_-$ for the sets of integers, positive integers and negative integers respectively. The set of complex numbers will be denoted by $\mathbf{C}$.

Let $\{e, f, h\}$ be the Chevalley basis of $sl(2, \mathbf{C})$, and $(\cdot, \cdot)$ the symmetric bilinear form defined by

$$(1.2.1) \qquad (x, y) = \mathrm{tr}(xy), \quad \text{for } x, y \in sl(2, \mathbf{C}).$$

Let $A = \mathbf{C}[s^{\pm 1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ be the ring of Laurent polynomials in commuting variables $s, t_1, \ldots, t_\nu$. For simplicity, we write $t^{\vec{n}} = t_1^{n_1} \cdots t_\nu^{n_\nu}$ for $\vec{n} = (n_1, \ldots, n_\nu) \in \mathbf{Z}^\nu$, $(\nu \geq 1)$. Then the toroidal Lie algebra of type A_1 is the vector space ([MRY][EM])

$$(1.2.2) \qquad \hat{\mathcal{G}} = sl(2, \mathbf{C}) \otimes A \oplus \Omega_A/dA,$$

where $\Omega_A/dA = \mathrm{span}\{\overline{s^m t^{\vec{n}}(t_i^{-1}dt_i)}, \overline{s^m t^{\vec{n}}(s^{-1}ds)}: \ m \in \mathbf{Z}, \vec{n} \in \mathbf{Z}^\nu, i = 1, 2, \ldots, \nu\}$, with the relations

$$(1.2.3) \qquad [x \otimes f_1, y \otimes f_2] = [x, y] \otimes f_1 f_2 + (x, y)\overline{f_2 df_1},$$

$$[\Omega_A/dA, \hat{\mathcal{G}}] = 0,$$

and

$$(1.2.4) \qquad m\overline{s^m t^{\vec{n}}(s^{-1}ds)} + \sum_{i=1}^{\nu} n_i \overline{s^m t^{\vec{n}}(t_i^{-1}dt_i)} = 0,$$

where $x, y \in sl(2, \mathbf{C})$, $f_1, f_2 \in A$ and $m \in \mathbf{Z}$, $\vec{n} = (n_1, \cdots, n_\nu) \in \mathbf{Z}^\nu$.

For simplicity, we write

$$(1.2.5) \qquad C_0(m, \vec{n}) = \overline{s^m t^{\vec{n}}(s^{-1}ds)}.$$

$$C_i(m,\vec{n}) = \overline{s^m t^{\vec{n}}(t_i^{-1}dt_i)},$$

for $m \in \mathbf{Z}, \vec{n} \in \mathbf{Z}^\nu, i = 1,2,\ldots,\nu$. Thus, we have

(1.2.6)
$$\overline{s^{m_2} t^{\vec{n}_2} d(s^{m_1} t^{\vec{n}_1})} = m_1 C_0(m_1+m_2, \vec{n}_1 + \vec{n}_2)$$
$$+\sum_{i=1}^{\nu} n_{1i} C_i(m_1+m_2, \vec{n}_1+\vec{n}_2),$$

for $m_i \in \mathbf{Z}, \vec{n}_i = (n_{i1},\ldots,n_{i\nu}) \in \mathbf{Z}^\nu, i = 1,2$. Thus, with this notation, (1.2.4) can expressed as follows

(1.2.4)′
$$\sum_{i=0}^{\nu} m_i C_i(m_0, \vec{m}) = 0,$$

where $m_0 \in \mathbf{Z}, \vec{m} = (m_1,\ldots,m_\nu) \in \mathbf{Z}^\nu$.

Let θ be an involution (i.e. autmorphism of order two [FLM]) of $sl(2,\mathbf{C})$ defined by

(1.2.7)
$$\theta(h) = h, \ \theta(e) = -e, \ \theta(f) = -f,$$

we put $x^\pm = f \pm e$, then $\{x^+, x^-, h\}$ form a basis of $sl(2,\mathbf{C})$ satisfying

(1.2.8)
$$\theta(h) = h, \ \ \theta(x^\pm) = -x^\pm,$$

and

(1.2.9)
$$[h, x^\pm] = -2x^\mp, \ \ [x^+, x^-] = 2h.$$

We extend θ to be an automorphism of $\hat{\mathcal{G}}$ by defining

(1.2.10)
$$\theta: \ \ x \otimes f_1(s,t_1,\ldots,t_\nu) \mapsto \theta(x) \otimes f_1(-s,t_1,\ldots,t_\nu),$$
$$C_j(m,\vec{n}) \mapsto (-1)^m C_j(m,\vec{n})$$

for $x \in sl(2,\mathbf{C})$, $f_1(s,t_1,\ldots,t_\nu) \in A$ and $m \in \mathbf{Z}, \vec{n} \in \mathbf{Z}^\nu, j = 0,1,\ldots,\nu$.

It is easy to see that the fixed-point set of θ forms a Lie subalgebra $\hat{\mathcal{G}}[\theta]$ of $\hat{\mathcal{G}}$, which indeed is spanned by the elements

(1.2.11)
$$\{x^\pm \otimes s^{2m+1} t^{\vec{n}}, h \otimes s^{2m} t^{\vec{n}}, C_j(2m,\vec{n}): \ m \in \mathbf{Z}, \vec{n} \in \mathbf{Z}^\nu, j = 0,1,\ldots,\nu\}.$$

Proposition 1.2.1 The subalgebra $\hat{\mathcal{G}}[\theta]$ is isomorphic to the toroidal Lie algebra $\hat{\mathcal{G}}$.

Proof. We define a map $\phi : \hat{\mathcal{G}} \to \hat{\mathcal{G}}[\theta]$ by

$$
\begin{gathered}
e \otimes s^m t^{\overrightarrow{n}} \mapsto e \otimes s^{2m+1} t^{\overrightarrow{n}}, \quad f \otimes s^m t^{\overrightarrow{n}} \mapsto f \otimes s^{2m-1} t^{\overrightarrow{n}}, \\
h \otimes s^m t^{\overrightarrow{n}} \mapsto h \otimes s^{2m} t^{\overrightarrow{n}} + C_0(2m, n), \\
C_0(m, \overrightarrow{n}) \mapsto 2C_0(2m, \overrightarrow{n}), \quad C_i(m, \overrightarrow{n}) \mapsto C_i(2m, \overrightarrow{n}),
\end{gathered}
\tag{1.2.12}
$$

for $m \in \mathbf{Z}, \overrightarrow{n} \in \mathbf{Z}^\nu, i = 1, 2, \ldots, \nu$. It is easy to check that ϕ defines an Lie algebra isomorphism from $\hat{\mathcal{G}}$ to $\hat{\mathcal{G}}[\theta]$.

□

We will identify the toroidal Lie algebra $\hat{\mathcal{G}}$ with its subalgebra $\hat{\mathcal{G}}[\theta]$ via this isomorphism ϕ. For convenience, we write

$$
\begin{gathered}
\alpha_1(2m+1, \overrightarrow{n}) := x^+ \otimes s^{2m+1} t^{\overrightarrow{n}}, \\
X(m, \overrightarrow{n}) = \begin{cases} x^- \otimes s^m t^{\overrightarrow{n}}, & \text{if } m \in 2\mathbf{Z}+1; \\ h \otimes s^m t^{\overrightarrow{n}}, & \text{if } m \in 2\mathbf{Z}, \end{cases}
\end{gathered}
\tag{1.2.13}
$$

for $m \in \mathbf{Z}, \overrightarrow{n} \in \mathbf{Z}^\nu$.

Thus the structure of $\hat{\mathcal{G}}[\theta]$ can be described as follows

$$
[\alpha_1(r_0, \overrightarrow{r}), X(n_0, \overrightarrow{n})] = 2X(r_0 + n_0, \overrightarrow{r} + \overrightarrow{n}), \tag{1.2.14}
$$

$$
[\alpha_1(r_0, \overrightarrow{R}), \alpha_1(s_0, \overrightarrow{s})] = 2\sum_{j=0}^{\nu} r_j C_j(r_0 + s_0, \overrightarrow{r} + \overrightarrow{s}), \tag{1.2.15}
$$

$$
\begin{gathered}
[X(m_0, \overrightarrow{m}), X(n_0, \overrightarrow{n})] \\
= \begin{cases} 2(-1)^{m_0+1}\alpha_1(m_0 + n_0, \overrightarrow{m} + \overrightarrow{n}), & \text{if } m_0 + n_0 \in 2\mathbf{Z}+1, \\ 2(-1)^{m_0}\sum_{j=0}^{\nu} m_j C_j(m_0 + n_0, \overrightarrow{m} + \overrightarrow{n}), & \text{if } m_0 + n_0 \in 2\mathbf{Z}, \end{cases}
\end{gathered}
\tag{1.2.16}
$$

and

$$
[C_j(2m_0, \overrightarrow{m}), \hat{\mathcal{G}}[\theta]] = 0. \tag{1.2.17}
$$

(1.2.18) $$2m_0C_0(2m_0, \overrightarrow{m}) + \sum_{i=1}^{\nu} m_iC_i(2m_0, \overrightarrow{m}) = 0,$$

where $r_0, s_0 \in 2\mathbf{Z}+1, m_0, n_0 \in \mathbf{Z}, \overrightarrow{r}, \overrightarrow{s}, \overrightarrow{m}, \overrightarrow{n} \in \mathbf{Z}^{\nu}$, and $\overrightarrow{r} = (r_1, \ldots, r_{\nu})$, $\overrightarrow{m} = (m_1, \ldots, m_{\nu})$.

It is clear that $\hat{\mathcal{G}}[\theta]$ has a natural $\mathbf{Z}^{\nu+1}$-gradation

(1.2.19) $$\hat{\mathcal{G}}[\theta] = \dot{+}_{m_0\in\mathbf{Z}, \overrightarrow{m}\in\mathbf{Z}^{\nu}} \mathcal{G}_{(m_0, \overrightarrow{m})},$$

where

$$\mathcal{G}_{(2m+1, \overrightarrow{n})} = \mathrm{span}\{X(2m+1, \overrightarrow{n}), \alpha_1(2m+1, \overrightarrow{n}) ;\ m \in \mathbf{Z}, \overrightarrow{n} \in \mathbf{Z}^{\nu}\},$$

$$\mathcal{G}_{(2m, \overrightarrow{n})} = \mathrm{span}\{X(2m, \overrightarrow{n}), C_j(2m, \overrightarrow{n});\ m \in \mathbf{Z}, \overrightarrow{n} \in \mathbf{Z}^{\nu}, j = 0, 1, \ldots, \nu\}.$$

Moreover, (1.2.14)-(1.2.17) give us

$$[\mathcal{G}_{(m_0, \overrightarrow{m})}, \mathcal{G}_{(n_0, \overrightarrow{n})}] \subset \mathcal{G}_{(m_0+n_0, \overrightarrow{m}+\overrightarrow{n})},$$

for $m_0, n_0 \in \mathbf{Z}, \overrightarrow{m}, \overrightarrow{n} \in \mathbf{Z}^{\nu}$.

Let $z, w, z_1, \ldots$ be formal variables. We define

(1.2.20) $$\alpha_1(z, \overrightarrow{n}) = \sum_{j\in 2\mathbf{Z}+1} \alpha_1(j, \overrightarrow{n})z^{-j},$$

$$X(z, \overrightarrow{n}) = \sum_{j\in\mathbf{Z}} X(j, \overrightarrow{n})z^{-j},$$

$$C_i(z, \overrightarrow{n}) = \sum_{j\in 2\mathbf{Z}} C_i(j, \overrightarrow{n})z^{-j},$$

for $i = 0, 1, \ldots, \nu$, $\overrightarrow{n} \in \mathbf{Z}^{\nu}$. Then the structure (1.2.14)-(1.2.18) of $\hat{\mathcal{G}}[\theta]$ can be rewritten compactly in terms of formal power series.

Lemma 1.2.2 The commutation relations (1.2.14)-(1.2.18) of $\hat{\mathcal{G}}[\theta]$ are equivalent to the following formal power series identities.

(1.2.21) $$[\alpha_1(z, \overrightarrow{m}), X(w, \overrightarrow{n})] = X(w, \overrightarrow{m}+\overrightarrow{n})(\delta(\frac{w}{z}) - \delta(-\frac{w}{z})),$$

(1.2.22)
$$[\alpha_1(z,\overrightarrow{m}),\alpha_1(w,\overrightarrow{n})] = C_0(w,\overrightarrow{m}+\overrightarrow{n})((D\delta)(\frac{w}{z}) - (D\delta)(-\frac{w}{z}))$$
$$+\sum_{j=1}^{\nu} m_j C_j(w,\overrightarrow{m}+\overrightarrow{n})(\delta(\frac{w}{z}) - \delta(-\frac{w}{z})).$$

(1.2.23)
$$[X(z,\overrightarrow{m}),X(w,\overrightarrow{n})] = 2C_0(w,\overrightarrow{m}+\overrightarrow{n})(D\delta)(-\frac{w}{z}) - 2(\alpha_1(w,\overrightarrow{m}+\overrightarrow{n})$$
$$-\sum_{j=1}^{\nu} m_j C_j(w,\overrightarrow{m}+\overrightarrow{n}))\delta(-\frac{w}{z}).$$

(1.2.24)
$$D_z C_0(z,\overrightarrow{m}) = \sum_{j=1}^{\nu} m_j C_j(z,\overrightarrow{m}).$$

for $\overrightarrow{m},\overrightarrow{n} \in \mathbf{Z}^{\nu}$, where $D_z = z\frac{\partial}{\partial z}$, and
$$\delta(z) = \sum_{i\in\mathbf{Z}} z^i, \quad (D\delta)(z) = \sum_{j\in\mathbf{Z}} iz^i.$$

Proof. The results of this Lemma follow by expanding the formal power series on both sides of the identities (1.2.21)-(1.2.24), and comparing the coefficients of $z^i w^j$, $i,j \in \mathbf{Z}$.

□

§1.3 The Heisenberg Algebra and Fock Space $\mathcal{M}$

Let $\Gamma_1 = \oplus_{i=1}^{\nu}(\mathbf{Z}c_i \oplus \mathbf{Z}d_i) \oplus \mathbf{Z}\alpha_1$ be a free additive group generated by the symbols $\alpha_1, c_i, d_i, i = 1, \ldots, \nu$. We define a non-degenerate symmetric bilinear form $(\cdot,\cdot) : \Gamma_1 \times \Gamma_1 \to \mathbf{Z}$ by

(1.3.1) $$(\alpha_1, \alpha_1) = 2, \quad (c_i, d_j) = \delta_{ij},$$

$$(\alpha_1, c_i) = (\alpha_1, d_i) = (c_i, c_j) = (d_i, d_j) = 0,$$

for $i, j = 1, 2, \ldots, \nu$.

Set $H = \mathbf{C} \otimes_{\mathbf{Z}} \Gamma_1$, and form a Lie algebra $\widetilde{\mathcal{H}}$, which is generated by the symbols $\alpha_1(m)$, $c_i(n), d_i(n)$ and c_0, $(i = 1, 2, \ldots, \nu, m \in 2\mathbf{Z} + 1, n \in 2\mathbf{Z})$, and the relations

(1.3.2) $$[\alpha_1(r), \alpha_1(s)] = r(\alpha_1, \alpha_1)\delta_{r+s,0}c_0,$$

$$[c_i(m), d_j(n)] = m(c_i, d_j)\delta_{m+n,0}c_0,$$

$$[\alpha_1(r), c_i(m)] = [\alpha_1(r), d_i(m)] = [c_i(m), c_j(m)] = [d_i(m), d_j(n)] = 0,$$

for $r, s \in 2\mathbf{Z} + 1$, $m, n \in 2\mathbf{Z}, i, j = 1, 2, \ldots, \nu$, and c_0 is central.

If we set $\alpha_1(2m) = c_i(2m + 1) = d_i(2m + 1) = 0$, for $m \in \mathbf{Z}$, we can express $\widetilde{\mathcal{H}}$ as follows

(1.3.3) $$\widetilde{\mathcal{H}} = (\oplus_{k\in\mathbf{Z}} H(k)) \oplus \mathbf{C}c_0,$$

where

$$H(k) = \mathbf{C}\alpha_1(k) \oplus \{\oplus_{1\le i\le\nu}(\mathbf{C}c_i(k) \oplus \mathbf{C}d_i(k))\}.$$

It is clear that

(1.3.4) $$\dim H(k) = \begin{cases} 2\nu, & \text{if } k \in 2\mathbf{Z}, \\ 1, & \text{if } k \in 2\mathbf{Z}+1, \end{cases}$$

and $\widetilde{\mathcal{H}}$ contains a Heisenberg subalgebra $\widehat{\mathcal{H}}$

(1.3.5) $$\widehat{\mathcal{H}} = \{\oplus_{k\in\mathbf{Z}\setminus\{0\}} H(k)\} \oplus \mathbf{C}c_0.$$

Let $\widehat{\mathcal{H}}^{\pm} = \oplus_{k\in\pm\mathbf{Z}_+} H(k)$. We know that the symmetric algebra $\mathcal{S}(\widehat{\mathcal{H}}_-)$ affords an $\widehat{\mathcal{H}}$-irreducible $\widetilde{\mathcal{H}}$-module. The action of $\widetilde{\mathcal{H}}$ on $\mathcal{S}(\widehat{\mathcal{H}}^-)$ is defined by saying

c_0 acts as k_0, ($k_0 \in \mathbf{C} \setminus \{0\}$).

$a(0)$ acts trivially,

$a(-m)$ acts as multiplication by $a(-m)$, and $a(m)$ acts as partial differential operator such that

$$a(m).1 = 0, \quad a(m).b(-n) = [a(m), b(-n)].1, \tag{1.3.6}$$

for $a, b \in H$, $m, n \in \mathbf{Z}_+$.

Now we form the Fock space

$$\mathcal{M} = \mathbf{C}[\Gamma_0] \otimes \mathcal{S}(\widehat{\mathcal{H}}^-), \tag{1.3.7}$$

where $\mathbf{C}[\Gamma_0]$ is the usual group algebra associated with the lattice $\Gamma_0 = \oplus_{1\leq i\leq \nu}(\mathbf{Z}c_i \oplus \mathbf{Z}d_i)$. Its base elements are of the form ϵ^{γ}, $\gamma \in \Gamma_0$.

We extend $\mathcal{M}$ to be both an $\widetilde{\mathcal{H}}$-module and a $\mathbf{C}[\Gamma_0]$-module by defining

$$a(m).\epsilon^{\gamma} \otimes u = \epsilon^{\gamma} \otimes a(m).u, \quad \text{for } m \in \mathbf{Z} \setminus \{0\}, \tag{1.3.8}$$

$$a(0).\epsilon^{\gamma} \otimes u = (a, \gamma)\epsilon^{\gamma} \otimes u,$$

$$\epsilon^{\mu}.\epsilon^{\gamma} \otimes u = \epsilon^{\mu+\gamma} \otimes u,$$

where $a \in H$, $\mu, \gamma \in \Gamma_0$ and $u \in \mathcal{S}(\widehat{\mathcal{H}}_-)$.

Remark. For simplicity, we will identify $\mathcal{H}$ with $\mathcal{H}(0)$.

§1.4 First Construction of $\widehat{\mathcal{G}}[\theta]$ by Vertex Operators

In this section, we will give a vertex operator representation of $\hat{\mathcal{G}}[\theta]$. The construction is a generalization of the level one representation of the affine Lie algebra $A_1^{(1)}$ by vertex operators [LW1][L1]. That is, we require that c_0 acts as $\frac{1}{2}$, and we want to extend the representation $\widehat{\mathcal{H}}$ on $\mathcal{M}$ to a representation of $\hat{\mathcal{G}}[\theta]$ on $\mathcal{M}$ by vertex operators.

For convenience, we set

$$c_0(n) = \delta_{n,0}c_0, \quad \text{for } n \in \mathbf{Z}. \tag{1.4.1}$$

For $\alpha \in \mathbf{Z}\alpha_1 \oplus (\oplus_{1\le i\le\nu}\mathbf{Z}c_i)$, $\beta \in \Gamma := \mathbf{Z}\alpha_1 + \sum_{1\le i\le\nu}(\mathbf{Z}c_i + \mathbf{Z}d_i) + \mathbf{Z}c_0$, we define our vertex operator

$$X_\beta(\alpha, z) := (-1)^{\frac{(\alpha,\alpha)}{2}} e^\alpha z^{2\alpha} E^-(\alpha, -z)\beta(-z)E^+(\alpha, -z), \tag{1.4.2}$$

where $\beta(z) = \sum_{j\in\mathbf{Z}} \beta(j)z^{-j}$, $\alpha \in \mathbf{Z}\alpha_1 \oplus (\oplus_{1\le i\le\nu}\mathbf{Z}c_i)$, and

$$E^\pm(\alpha, z) = \exp\Big(-\sum_{j\in\pm\mathbf{Z}_+} 2\alpha(j)z^{-j}/j\Big). \tag{1.4.3}$$

Here, the operators e^α, z^α are defined by

$$e^\alpha.(e^\gamma \otimes u) = e^{\gamma+\sum_{1\le i\le\nu}(\alpha,d_i)c_i} \otimes u, \tag{1.4.4}$$

$$z^\alpha.(e^\gamma \otimes u) = z^{(\alpha,\gamma)}e^\gamma \otimes u,$$

for $\alpha \in \mathbf{Z}\alpha_1 \oplus (\oplus_{1\le i\le\nu}\mathbf{Z}c_i)$, $e^\gamma \otimes u \in \mathcal{M}$.

If $X_\beta(\alpha, z)$ is expanded as a formal power series in z, then the coefficient of z^{-j}, written as $x_\beta(j, \alpha)$, is an operator acting on $\mathcal{M}$. Therefore, $X_\beta(\alpha, z)$ can be formally expanded as follows

$$X_\beta(\alpha, z) = \sum_{j\in\mathbf{Z}} x_\beta(j, \alpha)z^{-j}. \tag{1.4.5}$$

The following Lemma is clear

Lemma 1.4.1 For $\alpha \in \mathbf{Z}\alpha_1 \oplus (\oplus_{1\le i\le\nu}\mathbf{Z}c_i)$, $\beta_1, \beta_2 \in \Gamma$, and $n_1, n_2 \in \mathbf{Z}$. Then

$$X_{n_1\beta_1+n_2\beta_2}(\alpha, z) = n_1X_{\beta_1}(\alpha, z) + n_2X_{\beta_2}(\alpha, z), \tag{1.4.6}$$

or equivalently,

$$x_{n_1\beta_1+n_2\beta_2}(j, \alpha) = n_1x_{\beta_1}(j, \alpha) + n_2x_{\beta_2}(j, \alpha). \tag{1.4.7}$$

for $j \in \mathbf{Z}$.

□

Now, we state our main results

Theorem 1.4.2 The subspace, acting on $\mathcal{M}$, spanned by the operators

(1.4.8)

$$\{x_{\alpha_1}(2n+1, \sum_{1\le i\le \nu} m_i c_i),\ x_{c_0}(n.\alpha_1+\sum_{1\le i\le \nu} m_i c_i),\ x_{c_0}(2n, \sum_{1\le i\le \nu} m_i c_i),\ x_{c_j}(2n, \sum_{1\le i\le \nu} m_i c_i),\}$$

for $j = 1, 2, \ldots, \nu$, is a Lie algebar which is isomorphic to the toroidal Lie algebra $\hat{\mathcal{G}}[\theta]$ by the map ι given as follows.

(1.4.9)
$$x_{\alpha_1}(2n+1, \sum_{1\le i\le \nu} m_i c_i) \mapsto \alpha_1(2n+1, \vec{m}),$$
$$x_{c_0}(n.\alpha_1+\sum_{1\le i\le \nu} m_i c_i) \mapsto X(n, \vec{m}),$$
$$x_{c_0}(2n, \sum_{1\le i\le \nu} m_i c_i) \mapsto C_0(2n, \vec{m}),$$
$$x_{c_j}(2n, \sum_{1\le i\le \nu} m_i c_i) \mapsto C_j(2n, \vec{m}),$$

where $\vec{m} = (m_1, \ldots, m_\nu) \in \mathbf{Z}^\nu$, $n \in \mathbf{Z}$.

Theorem 1.4.3 The subspace, spanned by the operators $x_{d_j}(2n, \sum_{1\le i\le \nu} m_i c_i)$ together with the operators given in Theorem 1.4.2, is a Lie algebra which is isomorphic to the extended toroidal Lie algebra

(1.4.10)
$$\hat{\mathcal{G}}[\theta] \dotplus \mathcal{D},$$

where

(1.4.11)
$$\mathcal{D} = \operatorname{span}\{D_j(2n, \vec{m}) := s^{2n} t^{\vec{m}} (t_j \partial/\partial t_j) : \ n \in \mathbf{Z}, \vec{m} \in \mathbf{Z}^\nu, j = 1, \ldots, \nu\}.$$

The additional commutation relations in $\hat{\mathcal{G}}[\theta] \dotplus \mathcal{D}$ are given as follows (see [EM])

(1.4.12)
$$[D_i(m_0, \vec{m}), x \otimes f_1] = x \otimes D_i(m_0, \vec{m}).f_1,$$

(1.4.13)
$$[D_i(m_0, \vec{m}), C_j(n_0, \vec{n})]$$

$$= n_i C_j(m_0 + n_0, \vec{m} + \vec{n}) + \delta_{ij} \sum_{l=0}^{\nu} m_l C_l(m_0 + n_0, \vec{m} + \vec{n}).$$

$$(1.4.14) \qquad [D_i(m_0, \vec{m}), C_0(n_0, \vec{n})] = n_i C_0(m_0 + n_0, \vec{m} + \vec{n}).$$

$$(1.4.15) \quad [D_i(m_0, \vec{m}), D_j(n_0, \vec{n})] = n_i D_j(m_0 + n_0, \vec{m} + \vec{n}) - m_j D_i(m_0 + n_0, \vec{m} + \vec{n})$$
$$-n_i m_j \sum_{l=0}^{\nu} m_l C_l(m_0 + n_0, \vec{m} + \vec{n}).$$

for $m_0, n_0 \in 2\mathbf{Z}, x \otimes f_1 \in \hat{\mathcal{G}}[\theta]$, and $\vec{m} = (m_1, \ldots, m_\nu), \vec{n} = (n_1, \ldots, n_\nu) \in \mathbf{Z}^\nu$, $i, j = 1, 2, \ldots, \nu$. The isomorphism is defined by extending ψ from Theorem 1.4.2 as follows.

$$(1.4.16) \qquad x_{d_j}(2n, \sum_{i=1}^{\nu} m_i c_i) \mapsto D_j(2n, \vec{m}),$$

for $n \in \mathbf{Z}, \vec{m} = (m_1, \ldots, m_\nu) \in \mathbf{Z}^\nu$.

We will prove Theorem 1.4.2 and Theorem 1.4.3 in section 1.5 and section 1.6 respectively by a series of Lemmas.

§1.5 Proof of Theorem 1.4.2

Theorem 1.4.2 will be proved by the following a series of Lemmas. We first state a well-known result. Its proof can be found in [FLM].

Lemma 1.5.1 Let $Y(w, z)$ be a formal power series in w, z with coefficient in a vector space, such that $\lim_{z \to w} Y(w, z)$ exists (in the sense of [FLM]). Set $D_z = z\frac{\partial}{\partial z}$. Then

$$(1.5.1) \qquad Y(w, z)\delta(a\frac{w}{z}) = Y(w, aw)\delta(a\frac{w}{z}).$$

$$(1.5.2) \qquad Y(w, z)(D\delta)(a\frac{w}{z}) = Y(w, aw)(D\delta)(a\frac{w}{z}) + (D_z Y)(w, z)\delta(a\frac{w}{z}).$$

□

Lemma 1.5.2 Let $\alpha = \sum_{i=1}^{\nu} m_i c_i \in \dotplus_{1 \le j \le \nu} \mathbf{Z} c_j$. Then

$$(1.5.3) \qquad D_z X_{c_0}(\alpha, z) = X_\alpha(\alpha, z).$$

This is equivalent to

$$(1.5.3)' \qquad m_0 x_{c_0}(m_0,\alpha) + \sum_{i=1}^{\nu} m_i x_{c_i}(m_0,\alpha) = 0,$$

for $m_0 \in 2\mathbf{Z}, (m_1,\ldots,m_\nu) \in \mathbf{Z}^\nu$.

Proof. By Lemma 1.4.1, we have

$$X_\alpha(\alpha,z) = \sum_{i=1}^{\nu} m_i X_{c_i}(\alpha,z).$$

Thus one can check that (1.5.3) and (1.5.3)' are equivalent by expanding both sides of (1.5.3) into power series in z, and equating the coefficients of z^{m_0} for $m_0 \in \mathbf{Z}$.

To show (1.5.3), we note that $c_0(-z) = \frac{1}{2}$ (see (1.4.1)), and obtain

$$\begin{aligned}
D_z X_{c_0}(\alpha,z) &= D_z\{(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^\alpha z^{2\alpha}E^-(\alpha,-z)c_0(-z)E^+(\alpha,-z)\} \\
&= \frac{1}{2}(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^\alpha D_z\{z^{2\alpha}E^-(\alpha,-z)E^+(\alpha,-z)\} \\
&= \frac{1}{2}(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^\alpha D_z\{z^{2\alpha}\exp(\sum_{j\in 2\mathbf{Z}_+}\alpha(-j)z^j/j)\exp(-\sum_{j\in 2\mathbf{Z}_+}\alpha(j)z^{-j}/j)) \\
&= \frac{1}{2}(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^\alpha z^{2\alpha}E^-(\alpha,-z)\{2\alpha(0) + \sum_{j\in\mathbf{Z}_+}2\alpha(-2j)z^{2j} \\
&\qquad - \sum_{j\in\mathbf{Z}_+}(-2)\alpha(2j)z^{-2j}\}E^+(\alpha,-z) \\
&= (-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^\alpha z^{2\alpha}E^-(\alpha,-z)\alpha(-z)E^+(\alpha,-z) \\
&= X_\alpha(\alpha,z),
\end{aligned}$$

as required.

□

Lemma 1.5.3 Let $\alpha,\beta \in \mathbf{Z}\alpha_1 + (+_{1\le i\le\nu}\mathbf{Z}c_i)$, $\mu,\nu \in \Gamma$. Then

$$(1.5.4) \qquad z^\beta\epsilon^\alpha = \epsilon^\alpha z^\beta,\quad [\nu(z),\epsilon^\alpha] = \sum_{i=1}^{\nu}(\alpha,d_i)(\nu,c_i)\epsilon^\alpha.$$

$$(1.5.5) \qquad E^+(\alpha,z)E^-(\beta,w) = E^-(\beta,w)E^+(\alpha,z)(1-\frac{w}{z})^{(\alpha,\beta)}(1+\frac{w}{z})^{-(\alpha,\beta)}.$$

$$(1.5.6) \qquad [\mu(z), \nu(w)] = \sum_{j\in\mathbf{Z}} [\mu(j), \nu(-j)](\frac{w}{z})^j.$$

$$(1.5.7) \qquad [\nu(z), E^-(\alpha, w)] = 2E^-(\alpha, w) \sum_{j\in\mathbf{Z}_+} j^{-1}[\nu(j), \alpha(-j)](\frac{w}{z})^j.$$

$$(1.5.8) \qquad [E^+(\alpha, z), \nu(w)] = -2 \sum_{j\in\mathbf{Z}_+} j^{-1}[\alpha(j), \nu(-j)](\frac{w}{z})^j E^+(\alpha, z).$$

Proof. The first identity in (1.5.4) follows immediately from (1.4.4). To show the second one, we have, for $e^{\gamma} \otimes u \in \mathcal{M}$,

$$[\nu(z), e^{\alpha}].e^{\gamma} \otimes u = [\nu(0), e^{\alpha}].e^{\gamma} \otimes u$$

$$= (\nu(0)e^{\alpha} - e^{\alpha}\nu(0)).e^{\gamma} \otimes u$$

$$= \left(\nu, \gamma + \sum_{i=1}^{\nu}(\alpha, d_i)c_i\right) e^{\gamma+\sum_{i=1}^{\nu}(\alpha,d_i)c_i} \otimes u - (\nu, \gamma)e^{\gamma+\sum_{i=1}^{\nu}(\alpha,d_i)c_i} \otimes u$$

$$= \left(\sum_{i=1}^{\nu}(\alpha, d_i)(\nu, c_i)\right) e^{\gamma+\sum_{i=1}^{\nu}(\alpha,d_i)c_i} \otimes u$$

$$= \sum_{i=1}^{\nu}(\alpha, d_i)(\nu, c_i)e^{\alpha}.e^{\gamma} \otimes u,$$

as required.

For (1.5.5), we note that

$$[-\sum_{j\in\mathbf{Z}_+} 2\alpha(j)z^{-j}/j, \sum_{j\in\mathbf{Z}_+} 2\beta(-j)w^j/j]$$

$$= -\sum_{i,j\in\mathbf{Z}_+} \frac{4z^{-i}w^j}{ij}[\alpha(i), \beta(-j)]$$

$$= -\sum_{i,j\in 2\mathbf{N}+1} \frac{4z^{-i}w^j}{ij}(\alpha, \beta)i\delta_{i-j,0}c_0$$

$$= -2(\alpha, \beta) \sum_{j\in 2\mathbf{N}+1} (\frac{w}{z})^j/j$$

$$= \log(1 - \frac{w}{z})^{(\alpha,\beta)}(1 + \frac{w}{z})^{-(\alpha,\beta)}.$$

where the last equality follows from the identity [cf. [FLM]]

$$\log(1-\frac{w}{z})-\log(1+\frac{z}{w})=-2\sum_{j\in 2\mathbf{N}+1}\frac{1}{j}(\frac{w}{z})^j.$$

Thus (1.5.5) follows from this and the formal rule $e^Xe^Y=e^Ye^Xe^{[X,Y]}$, if $[X,Y]$ commutes with X and Y.

(1.5.6) is clear, and (1.5.7) (1.5.8) follow from the formal rule: $[X,e^Y]=[X,Y]e^Y$, if $[X,Y]$ commutes with X and Y, and the commutation relations

$$\begin{aligned}[\nu(z),\sum_{j\in\mathbf{Z}_+}2\alpha(-j)w^j/j]&=[\sum_{i\in\mathbf{Z}}\nu(i)z^{-i},\sum_{j\in\mathbf{Z}_+}2\alpha(-j)w^j/j]\\&=2\sum_{j\in\mathbf{Z}_+}j^{-1}[\nu(j),\alpha(-j)](\frac{w}{z})^j,\end{aligned}$$

and

$$\begin{aligned}[-\sum_{j\in\mathbf{Z}_+}2\alpha(j)z^{-j}/j,\nu(w)]&=[-\sum_{j\in\mathbf{Z}_+}2\alpha(j)z^{-j}/j,\sum_{i\in\mathbf{Z}}\nu(i)w^{-i}]\\&=-2\sum_{j\in\mathbf{Z}_+}j^{-1}[\alpha(j),\nu(-j)](\frac{w}{z})^i.\end{aligned}$$

This completes the proof of the Lemma.

□

To complete the proof of Theorem 1.4.2, we are required to check that the vertex operators $X_{\alpha_1}(\sum_{i=1}^{\nu}m_ic_i,z)$, $X_{c_0}(\alpha_1+\sum_{i=1}^{\nu}m_ic_i,z)$, $X_{c_0}(\sum_{i=1}^{\nu}m_ic_i,z)$, $X_{c_j}(\sum_{i=1}^{\nu}m_ic_i,z)$, for $\vec{m}=(m_1,\ldots,m_\nu)\in\mathbf{Z}^\nu$ (see (1.4.9) and (1.4.5)), satisfy the same commutation relations as in the identities (1.2.21)-(1.2.23).

Lemma 1.5.4 Let $\vec{m}=(m_1,\ldots,m_\nu),\vec{n}=(n_1,\ldots,n_\nu)\in\mathbf{Z}^\nu$, then

$$\begin{aligned}&[X_{\alpha_1}(\sum_{1\le i\le\nu}m_ic_i,z),X_{c_0}(\alpha_1+\sum_{1\le i\le\nu}n_ic_i,w)]\\&=X_{c_0}(\alpha_1+\sum_{1\le i\le\nu}(m_i+n_i)c_i,w)(\delta(\frac{w}{z})-\delta(-\frac{w}{z})).\end{aligned}\tag{1.5.9}$$

This gives the commutation relation corresponding to (1.2.21), or (1.2.14). That is

$$[x_{\alpha_1}(m_0,\sum_{1\le i\le\nu}m_ic_i),\ x_{c_0}(n_0,\alpha_1+\sum_{1\le i\le\nu}n_ic_i)]=2x_{c_0}(m_0+n_0,\alpha_1+\sum_{1\le i\le\nu}(m_i+n_i)c_i).$$

for $m_0 \in 2\mathbf{Z}+1$, $n_0 \in \mathbf{Z}$.

Proof. Set $\alpha = \sum m_i c_i$, $\beta = \alpha_1 + \sum n_i c_i$, and $\mu = \alpha_1$, $\nu = c_0$, then (1.5.9) can be written as follows

$$(1.5.10) \qquad [X_\mu(\alpha, z), X_\nu(\beta, w)] = X_{c_0}(\alpha+\beta, w)(\delta(\frac{w}{z}) - \delta(-\frac{w}{z})).$$

Note that $(\alpha, \beta) = 0$, and $\nu(-w) = \frac{1}{2}$, thus, by (1.5.4) and (1.5.5), we obtain

$$(1.5.11) \qquad X_\mu(\alpha, z)X_\nu(\beta, w)$$

$$= (-1)^{\frac{(\alpha,\alpha)}{2}+\frac{(\beta,\beta)}{2}} \epsilon^\alpha z^{2\alpha}\epsilon^\beta w^{2\beta} E^-(\alpha, -z)\mu(-z)E^+(\alpha, -z)E^-(\beta, -w)\nu(-w)E^+(\beta, -w)$$
$$= -\frac{1}{2}\epsilon^{\alpha+\beta} z^{2\alpha} w^{2\beta} E^-(\alpha, -z)\mu(-z)E^-(\beta, -w)E^+(\alpha, -z)E^+(\beta, -w),$$

where, by (1.5.7)

$$\mu(-z)E^-(\beta, -w) = [\mu(-z), E^-(\beta, -w)] + E^-(\beta, -w)\mu(-z)$$
$$= 2E^-(\beta, -w) \sum_{j\in\mathbf{Z}_+} j^{-1}[\mu(j), \beta(-j)](\frac{w}{z})^j + E^-(\beta, -w)\mu(-z)$$
$$= 2E^-(\beta, -w) \sum_{j\in 2\mathbf{N}+1} (\frac{w}{z})^j + E^-(\beta, -w)\mu(-z).$$

Similarly, we have

$$(1.5.12) \qquad X_\nu(\beta, w)X_\mu(\alpha, z)$$

$$= -\frac{1}{2}\epsilon^{\alpha+\beta} z^{2\alpha} w^{2\beta} E^-(\alpha, -z)E^-(\beta, -w)E^+(\beta, -w)\mu(-z)E^+(\alpha, -z),$$

where, by (1.5.8)

$$E^+(\beta, -w)\mu(-z) = [E^+(\beta, -w), \mu(-z)] + \mu(-z)E^+(\beta, -w)$$
$$= -2\sum_{j\in\mathbf{Z}_+} j^{-1}[\beta(j), \mu(-j)](\frac{z}{w})^j E^+(\beta, -w) + \mu(-z)E^+(\beta, -w)$$
$$= -2\sum_{j\in 2\mathbf{N}+1} (\frac{z}{w})^j E^+(\beta, -w) + \mu(-z)E^+(\beta, -w).$$

Therefore, (1.5.11) (1.5.12) give us

$$(1.5.13) \qquad [X_\mu(\alpha, z), X_\nu(\beta, w)] = X_\mu(\alpha, z)X_\nu(\beta, w) - X_\nu(\beta, w)X_\mu(\alpha, z)$$

$$= -\epsilon^{\alpha+\beta} z^{2\alpha} w^{2\beta} E^-(\alpha, -z)E^-(\beta, -w)E^+(\alpha, -z)E^+(\beta, -w)\{\sum_{j\in 2\mathbf{N}+1}(\frac{w}{z})^j + \sum_{j\in 2\mathbf{N}+1}(\frac{z}{w})^j\}$$

$$= -\frac{1}{2}\epsilon^{\alpha+\beta} z^{2\alpha} w^{2\beta} E^-(\alpha, -z)E^-(\beta, -w)E^+(\alpha, -z)E^+(\beta, -w)\{\delta(\frac{w}{z}) - \delta(-\frac{w}{z})\}.$$

By Lemma 1.5.1. and the fact that $E^{\pm}(\alpha, -z) = E^{\pm}(\alpha, z)$ for $\alpha \in \sum_{1\le i\le\nu} \mathbf{Z}c_i$, we obtain

$$[X_\mu(\alpha, z), X_\nu(\beta, w)]$$

$$= -\frac{1}{2}\epsilon^{\alpha+\beta} w^{2(\alpha+\beta)} E^-(\alpha+\beta, -w)E^+(\alpha+\beta, -w)\{\delta(\frac{w}{z}) - \delta(-\frac{w}{z})\}$$

$$= X_{c_0}(\alpha+\beta, w)(\delta(\frac{w}{z}) - \delta(-\frac{w}{z})),$$

since $(\alpha+\beta, \alpha+\beta) = 2$.

□

Lemma 1.5.5 Let $\vec{m} = (m_1, \dots, m_\nu), \vec{n} = (n_1, \dots, n_\nu) \in \mathbf{Z}^\nu$, then

$$(1.5.14) \qquad [X_{\alpha_1}(\sum_{1\le i\le\nu} m_i c_i, z), X_{\alpha_1}(\sum_{1\le i\le\nu} n_i c_i, w)]$$

$$= X_{c_0}(\sum_{1\le i\le\nu}(m_i+n_i)c_i, w)((D\delta)(\frac{w}{z}) - (D\delta)(-\frac{w}{z}))$$

$$+ \sum_{1\le j\le\nu} m_j X_{c_j}(\sum_{1\le i\le\nu}(m_i+n_i)c_i, w)(\delta(\frac{w}{z}) - \delta(-\frac{w}{z})).$$

This is the commutation relation corresponding to (1.2.22), or (1.2.15). That is

$$[x_{\alpha_1}(m_0, \sum_{1\le i\le\nu} m_i c_i), \; x_{\alpha_1}(n_0, \sum_{1\le i\le\nu} n_i c_i)] = 2\sum_{j=0}^{\nu} m_j x_{c_j}(m_0+n_0, \sum_{1\le i\le\nu}(m_i+n_i)c_i),$$

for $m_0, n_0 \in 2\mathbf{Z}+1$.

Proof. Let $\alpha = \sum_{1\le i\le\nu} m_i c_i, \beta = \sum_{1\le i\le\nu} n_i c_i$. Note that $(c_i, c_j) = 0$, $(\alpha_1, c_j) = 0$, for $i, j = 1, \dots, \nu$, we have

$$[X_{\alpha_1}(\alpha, z), X_{\alpha_1}(\beta, w)]$$

$$= [\epsilon^\alpha z^{2\alpha} E^-(\alpha, -z)\alpha_1(-z)E^+(\alpha, -z), \epsilon^\beta w^{2\beta} E^-(\beta, -w)\alpha_1(-w)E^+(\beta, -w)]$$

$$= \epsilon^{\alpha+\beta} z^{2\alpha} w^{2\beta} E^-(\alpha, -z)E^-(\beta, -w)[\alpha_1(-z), \alpha_1(-w)]E^+(\alpha, -z)E^+(\beta, -w).$$

where

$$[\alpha_1(-z), \alpha_1(-w)] = \sum_{j\in\mathbf{Z}}[\alpha_1(j), \alpha_1(-j)](\frac{w}{z})^j$$
$$= \sum_{j\in 2\mathbf{Z}+1} j(\frac{w}{z})^j = \frac{1}{2}((D\delta)(\frac{w}{z}) - (D\delta)(-\frac{w}{z})).$$

Thus, we have

$$[X_{\alpha_1}(\alpha, z), X_{\alpha_1}(\beta, w)] = \frac{1}{2}\epsilon^{\alpha+\beta} z^{2\alpha} w^{2\beta} E^-(\alpha, -z)E^-(\beta, -w)$$
$$\cdot E^+(\alpha, -z)E^+(\beta, -w)((D\delta)(\frac{w}{z}) - (D\delta)(-\frac{w}{z})).$$

Note that $(-z)^{2\alpha} = z^{2\alpha}$ and $E^\pm(\sum m_i c_i, -z) = E^\pm(\sum m_i c_i, z)$, so by Lemma 1.5.1 we obtain

$$[X_{\alpha_1}(\alpha, z), X_{\alpha_1}(\beta, w)]$$
$$= \frac{1}{2}\epsilon^{\alpha+\beta} w^{2(\alpha+\beta)} E^-(\alpha+\beta, -w)E^+(\alpha+\beta, -w)((D\delta)(\frac{w}{z}) - (D\delta)(-\frac{w}{z}))$$
$$+\frac{1}{2}\epsilon^{\alpha+\beta} w^{2(\alpha+\beta)} E^-(\alpha+\beta, -w)\{2\alpha(0) + \sum_{j\in 2\mathbf{Z}_+}(2\alpha(-j)w^j$$
$$+ \sum_{j\in 2\mathbf{Z}_+}(2\alpha(j)w^{-j}\}E^+(\alpha+\beta, -w)(\delta(\frac{w}{z}) - \delta(-\frac{w}{z}))$$
$$= X_{c_0}(\alpha+\beta, w)((D\delta)(\frac{w}{z}) - (D\delta)(-\frac{w}{z})) + \sum_{1\leq j\leq \nu} m_j X_{c_j}(\alpha+\beta, w)(\delta(\frac{w}{z}) - \delta(-\frac{w}{z})).$$

as required.

□

Lemma 1.5.6 Let $\vec{m} = (m_1, \ldots, m_\nu), \vec{n} = (n_1, \ldots, n_\nu) \in \mathbf{Z}^\nu$, then

$$[X_{c_0}(\alpha_1 + \sum_{1\leq i\leq \nu} m_i c_i, z), X_{c_0}(\alpha_1 + \sum_{1\leq i\leq \nu} n_i c_i, w)] \tag{1.5.15}$$
$$= 2X_{c_0}(\sum_{1\leq i\leq \nu}(m_i+n_i)c_i, w)(D\delta)(-\frac{w}{z}) - 2\{X_{\alpha_1}(\sum_{1\leq i\leq \nu}(m_i+n_i)c_i, w)$$
$$- \sum_{1\leq j\leq \nu} m_j X_{c_j}(\sum_{1\leq i\leq \nu}(m_i+n_i)c_i, w)\}\delta(-\frac{w}{z}).$$

This is the commutation relation corresponding to (1.2.23), or (1.2.16). That is

$$[x_{c_0}(m_0, \alpha_1 + \sum_{1\leq i\leq \nu} m_i c_i), \ x_{c_0}(n_0, \alpha_1 + \sum_{1\leq i\leq \nu} n_i c_i)]$$

$$= \begin{cases} 2(-1)^{m_0+1} x_{\alpha_1}(m_0+n_0, \sum_{1\le i\le\nu}(m_i+n_i)c_i), & \text{if } m_0+n_0 \in 2\mathbf{Z}+1, \\ 2(-1)^{m_0} \sum_{j=0}^{\nu} m_j x_{c_j}(m_0+n_0, \sum_{1\le i\le\nu}(m_i+n_i)c_i), & \text{if } m_0+n_0 \in 2\mathbf{Z}. \end{cases}$$

for $m_0, n_0 \in \mathbf{Z}$.

Proof. We set

$$\alpha = \alpha_1 + \sum_{1\le i\le\nu} m_i c_i, \quad \beta = \alpha_1 + \sum_{1\le i\le\nu} n_i c_i,$$

and note that $(\alpha, \beta) = 2$, we obtain, by applying (1.5.5),

$$[X_{c_0}(\alpha, z), X_{c_0}(\beta, w)]$$

$$= \frac{1}{4} e^{\alpha+\beta} z^{2\alpha} w^{2\beta} E^-(\alpha, -z) E^-(\beta, -w) E^+(\alpha, -z) E^+(\beta, -w)$$
$$\cdot \{(1-\frac{w}{z})^2(1+\frac{w}{z})^{-2} - (1-\frac{z}{w})^2(1+\frac{z}{w})^{-2}\},$$

where

$$(1-\frac{w}{z})^2(1+\frac{w}{z})^{-2} - (1-\frac{z}{w})^2(1+\frac{z}{w})^{-2} = 4(D\delta)(-\frac{w}{z}).$$

Therefore, by Lemma 1.5.1, we have

$$[X_{c_0}(\alpha, z), X_{c_0}(\beta, w)]$$

$$= e^{\alpha+\beta} w^{2(\alpha+\beta)} E^-(\alpha, w) E^-(\beta, -w) E^+(\alpha, w) E^+(\beta, -w)(D\delta)(-\frac{w}{z})$$
$$+ e^{\alpha+\beta} w^{2(\alpha+\beta)} E^-(\alpha, w) E^-(\beta, -w)\{2\alpha(0) + \sum_{j\in\mathbf{Z}_+} 2\alpha(-j) w^j + \sum_{j\in\mathbf{Z}_+} 2\alpha(j) w^{-j}\}$$
$$\cdot E^+(\alpha, w) E^+(\beta, -w)\delta(-\frac{w}{z}),$$

where

$$E^-(\alpha, w) E^-(\beta, -w)$$

$$= \exp(-\sum_{j\in\mathbf{Z}_-} 2\alpha(j) w^{-j}/j) \exp(-\sum_{j\in\mathbf{Z}_-} 2\beta(j)(-w)^{-j}/j)$$

$$= \exp\{\sum_{j\in\mathbf{Z}_+} \frac{2w^j}{j}(\alpha(-j) + (-1)^j \beta(-j))\}$$

$$= \exp\{\sum_{j\in 2\mathbf{Z}_+} \frac{2w^j}{j}(\sum_{1\le i\le\nu}(m_i+n_i)c_i(-j))\}$$

$$= E^-(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w),$$

where we have used the fact $\alpha(-j)+(-1)^j\beta(-j)=0$ for $j\in 2\mathbf{Z}+1$.

Similarly, we have

$$E^+(\alpha, w)E^+(\beta, -w) = E^+(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w).$$

Thus, we obtain

$$(1.5.16)\qquad [X_{c_0}(\alpha, z), X_{c_0}(\beta, w)]$$

$$= \epsilon^{\alpha+\beta}w^{2(\alpha+\beta)}E^-(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w)E^+(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w)(D\delta)(-\frac{w}{z})$$
$$+2\epsilon^{\alpha+\beta}w^{2(\alpha+\beta)}E^-(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w)\{-\alpha_1(-w)+\sum_{1\leq i\leq\nu}m_ic_i(-w)\}$$
$$\cdot E^+(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w)\delta(-\frac{w}{z}).$$

It is easy to see, as operator on $\mathcal{M}$, that (see (1.4.4))

$$\epsilon^{2\alpha_1+\sum_{1\leq i\leq\nu}(m_i+n_i)c_i} = \epsilon^{\sum_{1\leq i\leq\nu}(m_i+n_i)c_i},\quad w^{2\alpha_1+\sum_{1\leq i\leq\nu}(m_i+n_i)c_i} = w^{\sum_{1\leq i\leq\nu}(m_i+n_i)c_i}.$$

Therefore, we obtain

$$[X_{c_0}(\alpha, z), X_{c_0}(\beta, w)]$$
$$= \epsilon^{\sum_{1\leq i\leq\nu}(m_i+n_i)c_i}w^{2(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i)}E^-(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w)$$
$$\cdot E^+(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w)(D\delta)(-\frac{w}{z})$$
$$+2\epsilon^{\sum_{1\leq i\leq\nu}(m_i+n_i)c_i}w^{2(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i)}E^-(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w)$$
$$\cdot\{-\alpha_1(-w)+\sum_{1\leq i\leq\nu}m_ic_i(-w)\}E^+(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, -w)\delta(-\frac{w}{z})$$
$$= 2X_{c_0}(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, w)(D\delta)(-\frac{w}{z}) - 2\{X_{\alpha_1}(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, w)$$
$$-\sum_{1\leq j\leq\nu}m_jX_{c_j}(\sum_{1\leq i\leq\nu}(m_i+n_i)c_i, w)\}\delta(-\frac{w}{z}).$$

as required.

□

Now we finish the proof of Theorem 1.4.2. It is easy to see that the operators $x_{c_0}(2m_0, \sum_{1\leq i\leq \nu} m_i c_i)$, $x_{c_j}(2m_0, \sum_{1\leq i\leq \nu} m_i c_i)$ are central in the Lie algebra (spanned by the operators given in Theorem 1.4.2). Thus Theorem 1.4.2 follows from this and Lemma 1.5.2, Lemma 1.5.4-1.5.6.

□

§1.6 Proof of Theorem 1.4.3

Set $D_i(z, \vec{m}) = \sum_{j\in 2\mathbb{Z}} D_i(j, \vec{m}) z^{-j}$ for $\vec{m} \in \mathbb{Z}^\nu$, $i = 1, \ldots, \nu$, and $\alpha_1(z, \vec{m})$, $X(z, \vec{m})$, $C_i(z, \vec{m})$ are defined by (1.2.20). Then the commutation relations (1.4.12)-(1.4.15) are equivalent to the following formal power series identities

$$[D_i(z, \vec{m}), X(w, \vec{n})] = \frac{n_i}{2} X(w, \vec{m} + \vec{n})(\delta(\frac{w}{z}) + \delta(-\frac{w}{z})), \tag{1.6.1}$$

$$[D_i(z, \vec{m}), \alpha_1(w, \vec{n})] = \frac{n_i}{2} \alpha_1(w, \vec{m} + \vec{n})(\delta(\frac{w}{z}) + \delta(-\frac{w}{z})), \tag{1.6.2}$$

$$\begin{aligned}[D_i(z, \vec{m}), C_j(w, \vec{n})] &= \frac{1}{2}\delta_{ij} C_0(w, \vec{m} + \vec{n})((D\delta)(\frac{w}{z}) + (D\delta)(-\frac{w}{z})) \\ &+ \frac{1}{2}(n_i C_j(w, \vec{m} + \vec{n}) + \delta_{ij} \sum_{l=1}^{\nu} m_l C_l(w, \vec{m} + \vec{n}))(\delta(\frac{w}{z}) + \delta(-\frac{w}{z})),\end{aligned} \tag{1.6.3}$$

$$[D_i(z, \vec{m}), C_0(w, \vec{n})] = \frac{n_i}{2} C_0(w, \vec{m} + \vec{n})(\delta(\frac{w}{z}) + \delta(-\frac{w}{z})), \tag{1.6.4}$$

$$\begin{aligned}&[D_i(z, \vec{m}), D_j(w, \vec{n})] \\ &= \frac{1}{2}(n_i D_j(w, \vec{m} + \vec{n}) - m_j D_i(w, \vec{m} + \vec{n}))(\delta(\frac{w}{z}) + \delta(-\frac{w}{z})) \\ &\quad - \frac{n_i m_j}{2} \sum_{l=1}^{\nu} m_l C_l(w, \vec{m} + \vec{n})(\delta(\frac{w}{z}) + \delta(-\frac{w}{z})) \\ &\quad - \frac{n_i m_j}{2} C_0(w, \vec{m} + \vec{n})((D\delta)(\frac{w}{z}) + (D\delta)(-\frac{w}{z})),\end{aligned} \tag{1.6.5}$$

for $i, j = 1, \ldots, \nu$.

To show Theorem 1.4.3, we only need to prove that the corresponding vertex operators $X_{d_i}(\sum_{j=1}^{\nu} m_jc_j, z)$, $X_{\alpha_1}(\sum_{j=1}^{\nu} m_jc_j, z)$, $X_{c_0}(\alpha_1+\sum_{j=1}^{\nu} m_jc_j, z)$, $X_{c_0}(\sum_{j=1}^{\nu} m_jc_j, z)$, and $X_{c_i}(\sum_{j=1}^{\nu} m_jc_j, z)$ for $\vec{m} = (m_1, \dots, m_\nu) \in \mathbf{Z}^\nu$, [see (1.4.9) and (1.4.16)] satisfy the same commutation relations corresponding to the identities (1.6.1) -(1.6.5). We shall complete the proof of Theorem 1.4.3 by the following three Lemmas.

Lemma 1.6.1 Suppose $\alpha = \sum_{j=1}^{\nu} m_jc_j$, $\beta = \alpha_1 + \sum_{j=1}^{\nu} n_jc_j$ or $\sum_{j=1}^{\nu} n_jc_j$, and $\mu = d_i$, $\nu = c_0$ or α_1. Then we have

$$[X_\mu(\alpha, z), X_\nu(\beta, w)] = \frac{n_i}{2}X_\nu(\alpha + \beta, w)(\delta(\frac{w}{z}) + \delta(-\frac{w}{z})),$$

for $1 \le i \le \nu$. This gives the commutation relations corresponding to (1.6.1)(1.6.2) and (1.6.4), or (1.4.12)(1.4.14). That are

$$[x_{d_i}(m_0, \sum_{j=1}^{\nu} m_jc_j), x_{c_0}(n_0, \alpha_1 + \sum_{j=1}^{\nu} n_jc_j)] = n_ix_{c_0}(m_0 + n_0, \alpha_1 + \sum_{j=1}^{\nu}(m_j + n_j)c_j),$$

$$[x_{d_i}(m_0, \sum_{j=1}^{\nu} m_jc_j), x_{\alpha_1}(r_0, \sum_{j=1}^{\nu} r_jc_j)] = r_ix_{\alpha_1}(m_0 + r_0, \sum_{j=1}^{\nu}(m_j + r_j)c_j),$$

$$[x_{d_i}(m_0, \sum_{j=1}^{\nu} m_jc_j), x_{c_0}(k_0, \sum_{j=1}^{\nu} k_jc_j)] = k_ix_{\alpha_1}(m_0 + k_0, \sum_{j=1}^{\nu}(m_j + k_j)c_j),$$

where $m_0, k_0 \in 2\mathbf{Z}$, $r_0 \in 2\mathbf{Z} + 1$, $n_0, m_j, n_j, k_j, r_j \in \mathbf{Z}$, $j = 1, 2, \dots, \nu$.

Proof. As operators on $\mathcal{M}$, we have

$$[X_\mu(\alpha, z), X_\nu(\beta, w)] = X_\mu(\alpha, z)X_\nu(\beta, w) - X_\nu(\beta, w)X_\mu(\alpha, z),$$

and by (1.4.3)

$$X_\mu(\alpha, z)X_\nu(\beta, w) = (-1)^{\frac{(\alpha,\alpha)}{2}+\frac{(\beta,\beta)}{2}}e^\alpha z^{2\alpha}E^-(\alpha, -z)\mu(-z)E^+(\alpha, -z)$$

$$\cdot e^\beta w^{2\beta}E^-(\beta, -w)\nu(-w)E^+(\beta, -w).$$

Note that $E^\pm(\alpha, -z)$, and e^β, $w^{2\beta}$ act on different components of the Fock space $\mathcal{M}$, thus by (1.5.4) (1.5.5), we obtain

$$X_\mu(\alpha, z)X_\nu(\beta, w) = (-1)^{\frac{(\alpha,\alpha)}{2}+\frac{(\beta,\beta)}{2}}e^\alpha z^{2\alpha}w^{2\beta}E^-(\alpha, -z)\mu(-z)e^\beta E^-(\beta, -w)$$

$$\cdot\nu(-w)E^+(\alpha,-z)E^+(\beta,-w)$$

$$= (-1)^{\frac{(\alpha,\alpha)}{2}+\frac{(\beta,\beta)}{2}}\epsilon^{\alpha}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)\{[\mu(-z),\epsilon^{\beta}E^-(\beta,-w)]$$

$$+\epsilon^{\beta}E^-(\beta,-w)\mu(-z)\}\nu(-w)E^+(\alpha,-z)E^+(\beta,-w).$$

where

$$[\mu(-z),\epsilon^{\beta}E^-(\beta,-w)] = [\mu(-z),\epsilon^{\beta}]E^-(\beta,-w)+\epsilon^{\beta}[\mu(-z),E^-(\beta,-w)]$$

$$= \sum_{j=1}^{\nu}(\beta,d_j)(\mu,c_j)\epsilon^{\beta}E^-(\beta,-w)+\epsilon^{\beta}2E^-(\beta,-w)\sum_{j\in\mathbb{Z}_+}j^{-1}[\mu(j),\beta(-j)](\frac{w}{z})^j$$

$$= n_i\epsilon^{\beta}E^-(\beta,-w)+2\epsilon^{\beta}E^-(\beta,-w)\sum_{j\in 2\mathbb{Z}_+}\frac{n_i}{2}(\frac{w}{z})^j$$

$$= \epsilon^{\beta}E^-(\beta,-w)(n_i+n_i\sum_{j\in 2\mathbb{Z}_+}(\frac{w}{z})^j).$$

where we have used (1.5.4)(1.5.7) and (1.3.2). Note that $\mu(-z)\nu(-w) = \nu(-w)\mu(-z)$ [see (1.5.6)]. Thus we have

$$X_\mu(\alpha,z)X_\nu(\beta,w) \tag{1.6.6}$$

$$= (-1)^{\frac{(\alpha,\alpha)}{2}+\frac{(\beta,\beta)}{2}}\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\nu(-w)\mu(-z)E^+(\alpha,-z)E^+(\beta,-w)$$

$$+(-1)^{\frac{(\alpha+\beta,\alpha+\beta)}{2}}\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\nu(-w)E^+(\alpha,-z)E^+(\beta,-w)$$

$$\cdot\{n_i+n_i\sum_{j\in 2\mathbb{Z}_+}(\frac{w}{z})^j\}.$$

Similarly, we have

$$X_\nu(\beta,w)X_\mu(\alpha,z) = (-1)^{\frac{(\alpha+\beta,\alpha+\beta)}{2}}\epsilon^{\beta}w^{2\beta}E^-(\beta,-w)\nu(-w)E^+(\beta,-w)$$

$$\epsilon^{\alpha}z^{2\alpha}E^-(\alpha,-z)\mu(-z)E^+(\alpha,-z)$$

$$= (-1)^{\frac{(\alpha+\beta,\alpha+\beta)}{2}}\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\beta,-w)E^-(\alpha,-z)\nu(-w)E^+(\beta,-w)\mu(-z)E^+(\alpha,-z),$$

where

$$E^+(\beta,-w)\mu(-z) = [E^+(\beta,-w),\mu(-z)]+\mu(-z)E^+(\beta,-w)$$

$$= -2\sum_{j\in\mathbb{Z}_+}j^{-1}[\beta(j),\mu(-j)](\frac{z}{w})^jE^+(\beta,-w)+\mu(-z)E^+(\beta,-w)$$

$$= -\sum_{j\in 2\mathbf{Z}_+} n_i(\frac{z}{w})^j E^+(\beta,-w) + \mu(-z)E^+(\beta,-w).$$

here we have used (1.5.8) and (1.3.2). Thus

$$X_\nu(\beta,w)X_\mu(\alpha,z) \tag{1.6.7}$$

$$= (-1)^{\frac{(\alpha+\beta,\alpha+\beta)}{2}}\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\nu(-w)\mu(-z)E^+(\alpha,-z)E^+(\beta,-w)$$

$$-(-1)^{\frac{(\alpha+\beta,\alpha+\beta)}{2}}\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\nu(-w)E^+(\alpha,-z)E^+(\beta,-w)$$

$$\cdot\{n_i\sum_{j\in 2\mathbf{Z}_+}(\frac{z}{w})^j\}.$$

Therefore, (1.6.6) (1.6.7) give us

$$[X_\mu(\alpha,z),X_\nu(\beta,w)]$$

$$= \frac{n_i}{2}(-1)^{\frac{(\alpha+\beta,\alpha+\beta)}{2}}\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\nu(-w)E^+(\alpha,-z)E^+(\beta,-w)$$

$$\cdot(\delta(\frac{w}{z})+\delta(-\frac{w}{z})).$$

Finally, we note that $z^{2\alpha} = (-z)^{2\alpha}$, and $E^\pm(\alpha,-z) = E^\pm(\alpha,z)$ for $\alpha \in \sum_{1\le j\le\nu}\mathbf{Z}c_j$. Applying Lemma 1.5.1, we obtain

$$[X_\mu(\alpha,z),X_\nu(\beta,w)]$$

$$= \frac{n_i}{2}(-1)^{\frac{(\alpha+\beta,\alpha+\beta)}{2}}\epsilon^{\alpha+\beta}w^{2(\alpha+\beta)}E^-(\alpha+\beta,-w)\nu(-w)E^+(\alpha+\beta,-w)(\delta(\frac{w}{z})+\delta(-\frac{w}{z}))$$

$$= \frac{n_i}{2}X_\nu(\alpha+\beta,w)(\delta(\frac{w}{z})+\delta(-\frac{w}{z})).$$

as required.

□

Lemma 1.6.2 Suppose $\alpha = \sum_{i=1}^\nu m_ic_i$, $\beta = \sum_{i=1}^\nu n_ic_i$. Then

$$[X_{d_i}(\alpha,z),X_{c_j}(\beta,w)] = \frac{\delta_{ij}}{2}X_{c_0}(\alpha+\beta,w)((D\delta)(\frac{w}{z})+(D\delta)(-\frac{w}{z}))$$

$$+\frac{1}{2}\{n_iX_{c_j}(\alpha+\beta,w)+\delta_{ij}\sum_{l=1}^{\nu}m_lX_{c_l}(\alpha+\beta,w)\}(\delta(\frac{w}{z})+\delta(-\frac{w}{z}))$$

for $i, j \in \{1, \ldots, \nu\}$. This gives the commutation relation corrosponding to (1.6.3), or (1.4.13).

Proof. Since $(\alpha, \beta) = 0 = (\alpha, c_j)$, we have

$$X_{d_i}(\alpha, z)X_{c_j}(\beta, w)$$

$$= \epsilon^{\alpha} z^{2\alpha} E^-(\alpha, -z)d_i(-z)E^+(\alpha, -z)\epsilon^{\beta} w^{2\beta} E^-(\beta, -w)c_j(-w)E^+(\beta, -w)$$

$$= \epsilon^{\alpha} z^{2\alpha} w^{2\beta} E^-(\alpha, -z)d_i(-z)\epsilon^{\beta} E^-(\beta, -w)c_j(-w)E^+(\alpha, -z)E^+(\beta, -w),$$

where

$$d_i(-z)\epsilon^{\beta} E^-(\beta, -w)c_j(-w) \tag{1.6.8}$$

$$= [d_i(-z), \epsilon^{\beta} E^-(\beta, -w)c_j(-w)] + \epsilon^{\beta} E^-(\beta, -w)c_j(-w)d_i(-z)$$

$$= [d_i(-z), \epsilon^{\beta}]E^-(\beta, -w)c_j(-w) + \epsilon^{\beta}[d_i(-z), E^-(\beta, -w)]c_j(-w)$$

$$+\epsilon^{\beta} E^-(\beta, -w)[d_i(-z), c_j(-w)] + \epsilon^{\beta} E^-(\beta, -w)c_j(-w)d_i(-z).$$

We note that, by (1.5.4)(1.5.7) and (1.5.6)

$$[d_i(-z), \epsilon^{\beta}] = \sum_{l=1}^{\nu}(\beta, d_l)(d_i, c_l)\epsilon^{\beta} = n_i \epsilon^{\beta},$$

$$[d_i(-z), E^-(\beta, -w)] = 2E^-(\beta, -w) \sum_{l \in \mathbb{Z}_+} l^{-1}[d_i(l), \beta(-l)](\frac{w}{z})^l$$

$$= 2n_i E^-(\beta, -w) \sum_{l \in 2\mathbb{Z}_+} \frac{1}{2}(\frac{w}{z})^l = n_i E^-(\beta, -w) \sum_{l \in 2\mathbb{Z}_+} (\frac{w}{z})^l,$$

and

$$[d_i(-z), c_j(-w)] = \sum_{l \in \mathbb{Z}}[d_i(l), c_j(-l)](\frac{w}{z})^l$$

$$= \frac{1}{2}\delta_{ij} \sum_{l \in 2\mathbb{Z}} l(\frac{w}{z})^l.$$

Thus, (1.6.8) gives us

$$d_i(-z)\epsilon^{\beta} E^-(\beta, -w)c_j(-w)$$

$$= \epsilon^{\beta} E^-(\beta, -w)\{n_i c_j(-w) + n_i \sum_{l \in 2\mathbb{Z}_+} (\frac{w}{z})^l c_j(-w)$$

$$+\frac{\delta_{ij}}{2}\sum_{l\in 2\mathbf{Z}} l(\frac{w}{z})^l\} + \epsilon^{\beta} E^-(\beta,-w)c_j(-w)d_i(-z).$$

Therefore, we obtain from this and (1.6.8)

$$X_{d_i}(\alpha,z)X_{c_j}(\beta,w) \tag{1.6.9}$$

$$= \epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)c_j(-w)d_i(-z)E^+(\alpha,-z)E^+(\beta,-w)$$

$$+\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\{n_ic_j(-w) + n_i\sum_{l\in 2\mathbf{Z}_+}(\frac{w}{z})^l c_j(-w)$$

$$+\frac{\delta_{ij}}{2}\sum_{l\in 2\mathbf{Z}} l(\frac{w}{z})^l\}E^+(\alpha,-z)E^+(\beta,-w).$$

Similarly, we compute

$$X_{c_j}(\beta,w)X_{d_i}(\alpha,z)$$

$$= \epsilon^{\beta}w^{2\beta}E^-(\beta,-w)c_j(-w)E^+(\beta,-w)\epsilon^{\alpha}z^{2\alpha}E^-(\alpha,-z)d_i(-z)E^+(\alpha,-z)$$

$$= \epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\beta,-w)E^-(\alpha,-z)c_j(-w)E^+(\beta,-w)d_i(-z)E^+(\alpha,-z),$$

where, by (1.5.8)

$$E^+(\beta,-w)d_i(-z) = [E^+(\beta,-w),d_i(-z)] + d_i(-z)E^+(\beta,-w)$$

$$= -2\sum_{l\in\mathbf{Z}_+} l^{-1}[\beta(l),d_i(-l)](\frac{z}{w})^l E^+(\beta,-w) + d_i(-z)E^+(\beta,-w)$$

$$= -n_i\sum_{l\in 2\mathbf{Z}_+}(\frac{z}{w})^l E^+(\beta,-w) + d_i(-z)E^+(\beta,-w).$$

Thus, we have

$$X_{c_j}(\beta,w)X_{d_i}(\alpha,z) \tag{1.6.10}$$

$$= -\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\{n_i\sum_{l\in 2\mathbf{Z}_+}(\frac{z}{w})^l c_j(-w)\}$$

$$\cdot E^+(\alpha,-z)E^+(\beta,-w)$$

$$+\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)c_j(-w)d_i(-z)E^+(\alpha,-z)E^+(\beta,-w).$$

Therefore, (1.6.9) (1.6.10) give us

$$[X_{d_i}(\alpha,z),X_{c_j}(\beta,w)] = X_{d_i}(\alpha,z)X_{c_j}(\beta,w) - X_{c_j}(\beta,w)X_{d_i}(\alpha,z) \tag{1.6.11}$$

$$= n_i\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)c_j(-w)E^+(\alpha,-z)E^+(\beta,-w)\sum_{l\in 2\mathbf{Z}}(\frac{w}{z})^l$$

$$+\frac{\delta_{ij}}{2}\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)E^+(\alpha,-z)E^+(\beta,-w)\sum_{l\in 2\mathbf{Z}} l(\frac{w}{z})^l.$$

where

$$\sum_{l\in 2\mathbf{Z}}(\frac{w}{z})^l = \frac{1}{2}(\delta(\frac{w}{z})+\delta(-\frac{w}{z})).$$

$$\sum_{l\in 2\mathbf{Z}} l(\frac{w}{z})^l = \frac{1}{2}((D\delta)(\frac{w}{z})+(D\delta)(-\frac{w}{z})).$$

Applying Lemma 1.5.1, we obtain from (1.6.11)

$$[X_{d_i}(\alpha,z),X_{c_j}(\beta,w)]$$

$$= \frac{n_i}{2}\epsilon^{\alpha+\beta}w^{2(\alpha+\beta)}E^-(\alpha+\beta,-w)c_j(-w)E^+(\alpha+\beta,-w)(\delta(\frac{w}{z})+\delta(-\frac{w}{z}))$$

$$+\frac{\delta_{ij}}{4}\epsilon^{\alpha+\beta}w^{2(\alpha+\beta)}E^-(\alpha+\beta,-w)E^+(\alpha+\beta,-w)((D\delta)(\frac{w}{z})+(D\delta)(-\frac{w}{z}))$$

$$+\frac{\delta_{ij}}{2}\epsilon^{\alpha+\beta}w^{2(\alpha+\beta)}E^-(\alpha+\beta,-w)\{\sum_{l\in 2\mathbf{Z}}2\alpha(-l)w^l\}$$

$$\cdot E^+(\alpha+\beta,-w)(\delta(\frac{w}{z})+\delta(-\frac{w}{z})).$$

where

$$\sum_{l\in 2\mathbf{Z}}2\alpha(-l)w^l = 2\sum_{l=1}^{\nu}m_lc_l(-w).$$

Therefore, we obtain

$$[X_{d_i}(\alpha,z),X_{c_j}(\beta,w)] = \frac{n_i}{2}X_{c_j}(\alpha+\beta,w)(\delta(\frac{w}{z})+\delta(-\frac{w}{z}))$$

$$+\frac{\delta_{ij}}{2}X_{c_0}(\alpha+\beta,w)((D\delta)(\frac{w}{z})+(D\delta)(-\frac{w}{z}))$$

$$+\frac{\delta_{ij}}{2}\sum_{l=1}^{\nu}m_lX_{c_l}(\alpha+\beta,w)(\delta(\frac{w}{z})+\delta(-\frac{w}{z})).$$

as required.

□

Finally, we want to check the last commutation relation for $X_{d_i}(\alpha,z)$, $i=1,\dots,\nu$, $\alpha\in\sum_{j=1}^{\nu}\mathbf{Z}c_j$, which corresponds to the relation (1.6.5), or (1.4.15).

Lemma 1.6.3 Let $\alpha = \sum_{i=1}^{\nu} m_i c_i$, $\beta = \sum_{i=1}^{\nu} n_i c_i$, for $\vec{m} = (m_1, \ldots, m_\nu)$, $\vec{n} = (n_1, \ldots, n_\nu) \in \mathbf{Z}^\nu$. Then

$$
\begin{aligned}
(1.6.12) \qquad & [X_{d_i}(\alpha, z), X_{d_j}(\beta, w)] \\
&= \frac{1}{2}(n_i X_{d_j}(\alpha+\beta, w) - m_j X_{d_i}(\alpha+\beta, w))(\delta(\frac{w}{z}) + \delta(-\frac{w}{z})) \\
&\quad - \frac{n_i m_j}{2} \sum_{l=1}^{\nu} m_l X_{c_l}(\alpha+\beta, w)(\delta(\frac{w}{z}) + \delta(-\frac{w}{z})) \\
&\quad - \frac{n_i m_j}{2} X_{c_0}(\alpha+\beta, w)((D\delta)(\frac{w}{z}) + (D\delta)(-\frac{w}{z})).
\end{aligned}
$$

Proof. By a similar argument as in Lemma 1.6.1 and Lemma 1.6.2, we apply Lemma 1.5.3 to compute

$$
\begin{aligned}
(1.6.13) \qquad X_{d_i}(\alpha, z) X_{d_j}(\beta, w) &= (-1)^{\frac{\langle \alpha, \alpha \rangle}{2}} \epsilon^{\alpha} z^{2\alpha} E^-(\alpha, -z) d_i(-z) E^+(\alpha, -z) \\
&\quad \cdot (-1)^{\frac{\langle \beta, \beta \rangle}{2}} \epsilon^{\beta} w^{2\beta} E^-(\beta, -w) d_j(-w) E^+(\beta, -w) \\
&= \epsilon^{\alpha} z^{2\alpha} w^{2\beta} E^-(\alpha, -z) d_i(-z) \epsilon^{\beta} E^-(\beta, -w) E^+(\alpha, -z) d_j(-w) E^+(\beta, -w),
\end{aligned}
$$

where

$$
\begin{aligned}
d_i(-z)\epsilon^{\beta} E^-(\beta, -w) &= [d_i(-z), \epsilon^{\beta} E^-(\beta, -w)] + \epsilon^{\beta} E^-(\beta, -w) d_i(-z) \\
&= [d_i(-z), \epsilon^{\beta}] E^-(\beta, -w) + \epsilon^{\beta}[d_i(-z), E^-(\beta, -w)] + \epsilon^{\beta} E^-(\beta, -w) d_i(-z) \\
&= \sum_{l=1}^{\nu} (\beta, d_l)(d_i, c_l) \epsilon^{\beta} E^-(\beta, -w) + \epsilon^{\beta} 2 E^-(\beta, -w) \sum_{l \in \mathbf{Z}_+} l^{-1} [d_i(l), \beta(-l)] (\frac{w}{z})^l \\
&\quad + \epsilon^{\beta} E^-(\beta, -w) d_i(-z) \\
&= \epsilon^{\beta} E^-(\beta, -w) \{ n_i + n_i \sum_{l \in 2\mathbf{Z}_+} (\frac{w}{z})^l + d_i(-z) \}.
\end{aligned}
$$

and

$$
\begin{aligned}
E^+(\alpha, -z) d_j(-w) &= [E^+(\alpha, -z), d_j(-w)] + d_j(-w) E^+(\alpha, -z) \\
&= -2 \sum_{l \in \mathbf{Z}_+} l^{-1} [\alpha(l), d_j(-l)] (\frac{w}{z})^l + d_j(-w) E^+(\alpha, -z) \\
&= \{ -m_j \sum_{l \in 2\mathbf{Z}_+} (\frac{w}{z})^l + d_j(-w) \} E^+(\alpha, -z).
\end{aligned}
$$

Thus we obtain from (1.6.13)

$$(1.6.14)\qquad X_{d_i}(\alpha,z)X_{d_j}(\beta,w)$$
$$=\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\{n_i\sum_{l\in 2\mathbf{N}}(\frac{w}{z})^l+d_i(-z)\}$$
$$\cdot\{-m_j\sum_{l\in 2\mathbf{Z}_+}(\frac{w}{z})^l+d_j(-w)\}E^+(\alpha,-z)E^+(\beta,-w).$$

Similarly, as (1.6.14), we have

$$(1.6.15)\qquad X_{d_j}(\beta,w)X_{d_i}(\alpha,z)$$
$$=\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\{m_j\sum_{l\in 2\mathbf{N}}(\frac{z}{w})^l+d_j(-w)\}$$
$$\cdot\{-n_i\sum_{l\in 2\mathbf{Z}_+}(\frac{z}{w})^l+d_i(-z)\}E^+(\alpha,-z)E^+(\beta,-w).$$

We note that

$$(\sum_{l=0}^{\infty}x^{2l})(\sum_{l=1}^{\infty}x^{2l})-(\sum_{l=0}^{\infty}x^{-2l})(\sum_{l=1}^{\infty}x^{-2l})=\sum_{l\in\mathbf{Z}}lx^{2l},$$

and obtain, from this and (1.6.14), (1.6.15)

$$[X_{d_i}(\alpha,z),X_{d_j}(\beta,w)]$$
$$=\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\{-n_im_j\sum_{l\in\mathbf{Z}}l(\frac{w}{z})^{2l}+n_id_j(-w)\sum_{l\in 2\mathbf{Z}}(\frac{w}{z})^l$$
$$-m_jd_i(-z)\sum_{l\in 2\mathbf{Z}}(\frac{w}{z})^l+[d_i(-z),d_j(-w)]\}E^+(\alpha,-z)E^+(\beta,-w)$$
$$=-\frac{n_im_j}{4}\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)E^+(\alpha,-z)E^+(\beta,-w)\{(D\delta)(\frac{w}{z})+(D\delta)(-\frac{w}{z})\}$$
$$+\frac{1}{2}\epsilon^{\alpha+\beta}z^{2\alpha}w^{2\beta}E^-(\alpha,-z)E^-(\beta,-w)\{n_id_j(-w)-m_jd_i(-z)\}$$
$$\cdot E^+(\alpha,-z)E^+(\beta,-w)\{\delta(\frac{w}{z})+\delta(-\frac{w}{z})\}.$$

Since $d_i(w)=d_i(-w), c_i(w)=c_i(-w)$, and $E^\pm(\alpha,w)=E^\pm(\alpha,-w)$, we obtain, by applying Lemma 1.5.1

$$[X_{d_i}(\alpha,z),X_{d_j}(\beta,w)]$$
$$=\frac{1}{2}\epsilon^{\alpha+\beta}w^{2(\alpha+\beta)}E^-(\alpha+\beta,-w)\{n_id_j(-w)-m_jd_i(-w)\}$$

$$\cdot E^{+}(\alpha+\beta,-w)\{\delta(\frac{w}{z})+\delta(-\frac{w}{z})\}$$

$$-\frac{n_i m_j}{4}\epsilon^{\alpha+\beta}w^{2(\alpha+\beta)}E^{-}(\alpha+\beta,-w)E^{+}(\alpha+\beta,-w)\{(D\delta)(\frac{w}{z})+(D\delta)(-\frac{w}{z})\}$$

$$-\frac{n_i m_j}{4}\epsilon^{\alpha+\beta}w^{2(\alpha+\beta)}E^{-}(\alpha+\beta,-w)\{2\alpha(0)+2\sum_{l\in 2\mathbb{Z}_+}\alpha(l)w^{-l}+2\sum_{l\in -2\mathbb{Z}_+}\alpha(l)w^{-l}\}$$

$$\cdot E^{+}(\alpha+\beta,-w)\{\delta(\frac{w}{z})+\delta(-\frac{w}{z})\}$$

$$=\frac{1}{2}(n_i X_{d_j}(\alpha+\beta,w)-m_j X_{d_i}(\alpha+\beta,w))(\delta(\frac{w}{z})+\delta(-\frac{w}{z}))$$

$$-\frac{n_i m_j}{2}X_{c_0}(\alpha+\beta,w)((D\delta)(\frac{w}{z})+(D\delta)(-\frac{w}{z}))$$

$$-\frac{n_i m_j}{2}X_{\alpha}(\alpha+\beta,w)(\delta(\frac{w}{z})+\delta(-\frac{w}{z})).$$

where, by Lemma 1.4.1

$$X_{\alpha}(\alpha+\beta,w)=\sum_{l=1}^{\nu}m_l X_{c_l}(\alpha+\beta,w),$$

and which completes the proof of the Lemma, and also finished the proof of Theorem 1.4.3.

□

§1.7 Structure of $\mathcal{M}$

Recall that $\Gamma_0=\sum_{i=1}^{\nu}(\mathbb{Z}c_i+\mathbb{Z}d_i)$, and $\mathcal{M}=\mathbb{C}[\Gamma_0]\otimes S(\mathcal{H}^-)$, where $\mathcal{H}^-=\oplus_{k\in\mathbb{Z}_-}H(k)$ and H is the $\mathbb{C}$-vector space with basis α_1,c_i,d_i, $i=1,\ldots,\nu$.

We define Segal operator L_n $(n\in\mathbb{Z})$ on $\mathcal{M}$ by

$$\text{(1.7.1)}\qquad L_n=\frac{1}{2}\sum_{j\in\mathbb{Z}}:\alpha_1(n-j)\alpha_1(j):+2\sum_{j\in\mathbb{Z}}\sum_{i=1}^{\nu}:c_i(n-j)d_i(j):$$

for $n\in\mathbb{Z}$, where $:\ :$ is the so called normal ordering defined by

$$:a(i)b(j):=\begin{cases}a(i)b(j), & \text{if } i\le j,\\ b(j)a(i), & \text{if } j<i,\end{cases}$$

for $a,b\in H$, $i,j\in\mathbb{Z}$.

Since $\alpha_1(2n) = c_i(2n+1) = d_i(2n+1) = 0$ for all $n \in \mathbb{Z}$, we have

$$L_{2n+1} = 0, \quad \text{for } n \in \mathbb{Z}. \tag{1.7.2}$$

In next Lemma, we will apply the technique from [KR].

Lemma 1.7.1 For $\alpha \in H$, and $m \in \mathbb{Z}, n \in 2\mathbb{Z}$, we have

$$[\alpha(m), L_n] = m\alpha(m+n). \tag{1.7.3}$$

Proof. Let χ be the real function defined by

$$\chi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

We define

$$L_n(\epsilon) = \frac{1}{2}\sum_{j\in\mathbb{Z}} :\alpha_1(n-j)\alpha_1(j): \chi(j\epsilon) + 2\sum_{j\in\mathbb{Z}}\sum_{i=1}^{\nu} :c_i(n-j)d_i(j): \chi(j\epsilon) \tag{1.7.4}$$

for $\epsilon > 0$.

Note that $L_n(\epsilon)$ is a truncted operator for $\epsilon > 0$(that is, the summation in (1.7.4) are finite), and we have $\lim_{\epsilon\to 0} L_n(\epsilon) = L_n$. The limit means that, for any given $x \in \mathcal{M}$, $L_n(\epsilon).x = L_n.x$ provided $\epsilon > 0$ and sufficiently small.

To prove this Lemma, we are going to check (1.7.3) for $\alpha = \alpha_1$, c_i, or d_i, $i = 1, \ldots, \nu$. First, if $\alpha = \alpha_1$, we only need to consider the case for $m \in 2\mathbb{Z}+1$. By (1.3.1) and (1.3.2), we have

$$[\alpha_1(m), L_n(\epsilon)] = [\alpha_1(m), \frac{1}{2}\sum_{j\in\mathbb{Z}} :\alpha_1(n-j)\alpha_1(j): \chi(j\epsilon)]$$

$$= \frac{1}{2}\sum_{j\in 2\mathbb{Z}+1} [\alpha_1(m), :\alpha_1(n-j)\alpha_1(j):]\chi(j\epsilon)$$

$$= \frac{1}{2}\sum_{j\in 2\mathbb{Z}+1} \{[\alpha_1(m), \alpha_1(n-j)]\alpha_1(j) + \alpha_1(n-j)[\alpha_1(m), \alpha_1(j)]\}\chi(j\epsilon)$$

$$= \frac{1}{2}\sum_{j\in 2\mathbb{Z}+1} \{m\delta_{m+n-j,0}\alpha_1(j) + \alpha_1(n-j)m\delta_{m+j,0}\}\chi(j\epsilon)$$

$$= \frac{1}{2} m\alpha_1(m+n)\{\chi((m+n)\epsilon) + \chi(-m\epsilon)\}.$$

Let $\epsilon \to 0$, then the above identities imply

$$[\alpha_1(m), \quad L_n] = m\alpha_1(m+n),$$

as required.

Now, we let $\alpha = c_l$, for some $1 \le l \le \nu$, so we may assume $m \in 2\mathbf{Z}$.

$$[c_l(m), L_n(\epsilon)] = [c_l(m), 2\sum_{j\in\mathbf{Z}}\sum_{i=1}^{\nu} : c_i(n-j)d_i(j) : \chi(j\epsilon)]$$

$$= 2\sum_{j\in\mathbf{Z}}\sum_{i=1}^{\nu}[c_l(m), : c_i(n-j)d_i(j) :]\chi(j\epsilon)$$

$$= 2\sum_{j\in\mathbf{Z}}[c_l(m), : c_l(n-j)d_l(j) :]\chi(j\epsilon)$$

$$= 2\sum_{j\in\mathbf{Z}} c_l(n-j)[c_l(m), d_l(j)]\chi(j\epsilon)$$

$$= 2\sum_{j\in\mathbf{Z}} c_l(n-j)m\frac{1}{2}\delta_{m+j,0}\chi(j\epsilon) = mc_l(m+n)\chi(-m\epsilon).$$

Therefore, letting $\epsilon \to 0$, we obtain

$$[c_l(m), \quad L_n] = mc_l(m+n),$$

as required. Finally, for $\alpha = d_l$, $(l = 1, \ldots, \nu)$, we can also prove $[d_l(m), \quad L_n] = md_l(m+n)$, by a similar argument to the above case.

□

Lemma 1.7.2 Let $b_1, \ldots, b_k \in H$, $\mu \in \Gamma_0 = \sum_{i=1}^{\nu}(\mathbf{Z}c_i + \mathbf{Z}d_i)$. We have

$$\text{(1.7.5)} \qquad L_0.(e^{\mu} \otimes b_1(-m_1)\cdots b_k(-m_k))$$

$$= \left((\sum_{i=1}^{k} m_i) + (\mu, \mu)\right)(e^{\mu} \otimes b_1(-m_1)\cdots b_k(-m_k)),$$

for $m_1, \ldots, m_k \in \mathbf{Z}_+$.

Proof. First, we compute

$$L_0.(e^{\mu} \otimes 1) = 2\sum_{j\in\mathbf{Z}}\sum_{i=1}^{\nu} : c_i(-j)d_i(j) : (e^{\mu} \otimes 1)$$

$$= 2\sum_{i=1}^{\nu} : c_i(0)d_i(0) : (e^{\mu} \otimes 1)$$

$$= 2\sum_{i=1}^{\nu}(c_i, \mu)(d_i, \mu)(e^{\mu} \otimes 1) = (\mu, \mu)e^{\mu} \otimes 1.$$

Thus, applying this and Lemma 1.7.1, we obtain

$$L_0.(e^{\mu} \otimes b_1(-m_1)\cdots b_k(-m_k))$$

$$= \{[L_0, b_1(-m_1)\cdots b_k(-m_k)] + b_1(-m_1)\cdots b_k(-m_k)L_0\}.e^{\mu} \otimes 1$$
$$= \left((\sum_{l=1}^{k} m_l) + (\mu, \mu)\right) b_1(-m_1)\cdots b_k(-m_k).e^{\mu} \otimes 1$$
$$= \left((\sum_{l=1}^{k} m_l) + (\mu, \mu)\right) e^{\mu} \otimes b_1(-m_1)\cdots b_k(-m_k),$$

as required.

□

We now use (1.7.5) to define a $\mathbf{Z}^{2\nu+1}$-grading on $\mathcal{M}$ by assigning the degree of $e^{\mu} \otimes b_1(-m_1)\cdots b_k(-m_k)$ to be $(\sum_{l=1}^{k} m_l + (\mu, \mu), \ \mu)$ for $\mu \in \Gamma_0$, $m_1, \ldots, m_k \in \mathbf{Z}_+$ and $b_1, \ldots, b_k \in H$.

Proposition 1.7.3 Let $\alpha \in \mathbf{Z}\alpha_1 + \sum_{1\le i\le \nu} \mathbf{Z}c_i$, $\beta \in \Gamma = \mathbf{Z}\alpha_1 + \sum_{1\le i\le \nu}(\mathbf{Z}c_i + \mathbf{Z}d_i)$ $+\mathbf{Z}c_0$. Then the component $x_\beta(m, \alpha)$ is an operator of $\mathcal{M}$ of degree $(-m, \sum_{i=1}^{\nu}(\alpha, d_i)c_i)$ for $m \in \mathbf{Z}$.

Proof. It is clear that $\alpha(m)$ acts on $\mathcal{M}$ with degree $(-m, 0)$ for $\alpha \in H$, $m \in \mathbf{Z}$. Thus, if we define the operator $D_p(\alpha, \beta) \in \mathrm{End}\mathcal{M}$ by

$$E^-(\alpha, -z)\beta(-z)E^+(\alpha, -z) = \sum_{p\in\mathbf{Z}} D_p(\alpha, \beta)z^p.$$

We see that $D_p(\alpha, \beta)$ acts on $\mathcal{M}$ with degree $(p, 0)$.

Let $e^{\mu} \otimes u$ be an element of $\mathcal{M}$ with homogeneous degree $(n + (\mu, \mu), \ \mu)$. We have

$$\sum_{m\in\mathbf{Z}} x_\beta(m, \alpha)z^{-m}.(e^{\mu} \otimes u) = X_\beta(\alpha, z).e^{\mu} \otimes u$$

$$= (-1)^{\frac{(\alpha,\alpha)}{2}} e^{\alpha} z^{2\alpha} E^-(\alpha, -z)\beta(-z)E^+(\alpha, -z).(e^{\mu} \otimes u)$$

$$= (-1)^{\frac{(\alpha,\alpha)}{2}} \sum_{p\in\mathbf{Z}} e^{\alpha} z^{2\alpha} D_p(\alpha,\beta).(e^{\mu} \otimes u) z^{p}$$

$$= (-1)^{\frac{(\alpha,\alpha)}{2}} \sum_{p\in\mathbf{Z}} (D_p(\alpha,\beta).e^{\mu+\sum_{1\le i\le\nu}(\alpha,d_i)c_i} \otimes u) z^{p+2(\alpha,\mu)}$$

$$= (-1)^{\frac{(\alpha,\alpha)}{2}} \sum_{m\in\mathbf{Z}} (D_{-m-2(\alpha,\mu)}(\alpha,\beta).e^{\mu+\sum_{1\le i\le\nu}(\alpha,d_i)c_i} \otimes u) z^{-m}.$$

Note that, for $\alpha \in \mathbf{Z}\alpha_1 + \sum_{1\le i\le\nu} \mathbf{Z}c_i$, $\mu \in \sum_{1\le i\le\nu}(\mathbf{Z}c_i + \mathbf{Z}d_i)$, we have

$$(\alpha,\mu) = \sum_{1\le i\le\nu} (\alpha,d_i)(\mu,c_i).$$

Thus, the element $D_{-m-2(\alpha,\mu)}(\alpha,\beta).e^{\mu+\sum_{1\le i\le\nu}(\alpha,d_i)c_i} \otimes u$ of $\mathcal{M}$ has degree

$$(-m-2(\alpha,\mu),0)$$

$$+\left(n + (\mu + \sum_{1\le i\le\nu}(\alpha,d_i)c_i, \mu + \sum_{1\le i\le\nu}(\alpha,d_i)c_i),\ \mu + \sum_{1\le i\le\nu}(\alpha,d_i)c_i\right)$$

$$= (-m-2(\alpha,\mu),0) + \left(n + (\mu,\mu) + 2\sum_{1\le i\le\nu}(\alpha,d_i)(\mu,c_i),\ \mu + \sum_{1\le i\le\nu}(\alpha,d_i)c_i\right)$$

$$= \left(-m + n + (\mu,\mu), \mu + \sum_{1\le i\le\nu}(\alpha,d_i)c_i\right)$$

$$= (n + (\mu,\mu),\mu) + \left(-m, \sum_{1\le i\le\nu}(\alpha,d_i)c_i\right),$$

which implies that the operator $x_\beta(m,\alpha)$ has degree $(-m,\sum_{1\le i\le\nu}(\alpha,d_i)c_i)$, as required.

$\square$

Corollary 1.7.4 Let $\alpha \in \mathbf{Z}\alpha_1 + \sum_{1\le i\le\nu}\mathbf{Z}c_i$, $\beta \in \Gamma$. Then the component operator $x_\beta(m,\alpha)$ are locally nilpotently acting on $\mathcal{M}$ for $m \in \mathbf{Z}_+$.

Proof. Since there is no non-zero element of $\mathcal{M}$ with degree of the form $(-p,\mu)$ for $p \in \mathbf{Z}_+$, $\mu \in \sum_{1\le i\le\nu}(\mathbf{Z}c_i + \mathbf{Z}d_i)$. The result follows from the fact that $x_\beta(m,\alpha)$ is an operator of $\mathcal{M}$ with degree $(-m,\sum_{1\le i\le\nu}(\alpha,d_i)c_i)$.

$\square$

Let $\lambda \in \Gamma_0 = \sum_{1\leq i\leq \nu}(\mathbf{Z}c_i + \mathbf{Z}d_i)$ be fixed. We define

$$(1.7.6) \qquad \mathcal{M}(\lambda) := \epsilon^{\lambda+\sum_{1\leq i\leq\nu}\mathbf{Z}c_i} \otimes \mathcal{S}(\mathcal{H}^-).$$

Then the component operator $x_\beta(m,\alpha)$, $(m \in \mathbf{Z})$, of the vertex operator $X_\beta(\alpha, z)$, $(\alpha \in \mathbf{Z}\alpha_1 + \sum_{1\leq i\leq\nu}\mathbf{Z}c_i,\ \beta \in \Gamma)$ acts on $\mathcal{M}(\lambda)$, and $\mathcal{M}(\lambda)$ is a $\hat{\mathcal{G}}[\theta]$ (and $\tilde{\mathcal{G}}[\theta]$) submodule of $\mathcal{M}$.

Let $\lambda_i := (\lambda, c_i)$, $i = 1, \ldots, \nu$. For $\alpha \in \mathbf{Z}\alpha_1 + \sum_{1\leq i\leq\nu}\mathbf{Z}c_i$, we define $S_p(\alpha)$, $p \in \mathbf{Z}$ by

$$(1.7.7) \qquad E^-(\alpha, -z) = \sum_{p=0}^{\infty} S_p(\alpha)z^p,$$

and $S_p(\alpha) = 0$ for $p < 0$.

We have

$$\begin{aligned}
\sum_{m\in\mathbf{Z}} x_{c_0}(m,\alpha)z^{-m}.(\epsilon^\lambda \otimes 1) &= X_{c_0}(\alpha, z).(\epsilon^\lambda \otimes 1)\\
&= \frac{1}{2}(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^\alpha z^{2\alpha}E^-(\alpha,-z).\epsilon^\lambda \otimes 1\\
&= \frac{1}{2}(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^\alpha z^{2\alpha}\sum_{p=0}^{\infty} S_p(\alpha).(\epsilon^\lambda \otimes 1)z^p\\
&= \frac{1}{2}(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^{\lambda+\sum_{1\leq i\leq\nu}(\alpha,d_i)c_i} \otimes \sum_{p=0}^{\infty} S_p(\alpha)z^{p+2(\alpha,\lambda)}\\
&= \frac{1}{2}(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^{\lambda+\sum_{1\leq i\leq\nu}(\alpha,d_i)c_i} \otimes \sum_{m\in\mathbf{Z}} S_{-m-2(\alpha,\lambda)}z^{-m}.
\end{aligned}$$

This gives us

$$(1.7.8) \qquad x_{c_0}(m,\alpha).(\epsilon^\lambda \otimes 1) = \frac{1}{2}(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^{\lambda+\sum_{1\leq i\leq\nu}(\alpha,d_i)c_i} \otimes S_{-m-2(\alpha,\lambda)}(\alpha).$$

In particular, we have

$$(1.7.9) \qquad x_{c_0}(-2(\alpha,\lambda),\alpha).(\epsilon^\lambda \otimes 1) = \frac{1}{2}(-1)^{\frac{(\alpha,\alpha)}{2}}\epsilon^{\lambda+\sum_{1\leq i\leq\nu}(\alpha,d_i)c_i} \otimes 1.$$

$$x_{c_0}(m,\alpha).(\epsilon^\lambda \otimes 1) = 0, \text{ if } m > -2(\alpha,\lambda).$$

and

$$(1.7.10)\quad x_{c_0}(m.\sum_{1\le i\le\nu}k_ic_i).(e^{\lambda}\otimes 1)=\frac{1}{2}e^{\lambda+\sum_{1\le i\le\nu}k_ic_i}\otimes S_{-m-2\sum_{1\le i\le\nu}k_iX_i}(\sum_{1\le i\le\nu}k_ic_i).$$

Proposition 1.7.5 (i) $\mathcal{M}(\lambda)$ is an irreducible $\tilde{\mathcal{G}}[\theta]$-module. for $\lambda\in\Gamma_0$.

(ii) $\mathcal{M}$ is a completely reducible $\tilde{\mathcal{G}}[\theta]$-module. and

$$\mathcal{M}=\dotplus_{\substack{n_i\in\mathbf{Z}\\1\le i\le\nu}}\mathcal{M}(\sum_{1\le i\le\nu}n_id_i).$$

Proof. Since $\mathcal{S}(\mathcal{H}^-)$ is an irreducible $\mathcal{H}$-module. (i) follows from this and (1.7.9). (ii) follows from (i).

□

Let $\pi_\lambda:\ \tilde{\mathcal{G}}[\theta]\to\mathrm{End}\mathcal{M}(\lambda)$ be the representation afforded by the $\tilde{\mathcal{G}}[\theta]$-module $\mathcal{M}(\lambda)$. Define $\mathrm{Ker}(\pi_\lambda)$:=the kernel of π_λ.

Theorem 1.7.6 (i) The dimension of the kernel $\mathrm{Ker}(\pi_\lambda)$ of the representation of $\tilde{\mathcal{G}}[\theta]$ on $\mathcal{M}(\lambda)$ is ν . and $\mathrm{Ker}(\pi_\lambda)$ is spanned by the operators

$$K_i(\lambda):=c_i(0)-2(\lambda.c_i)c_0.$$

for $i=1.2.\ldots.\nu$.

(ii) The Lie algebra $\tilde{\mathcal{G}}[\theta]$ is faithfully representated on $\mathcal{M}$.

Proof. One can easily check that $K_i(\lambda)\in\mathrm{Ker}(\pi_\lambda)$. $1\le i\le\nu$. To show that $K_1(\lambda).K_2(\lambda).\ldots.K_\nu(\lambda)$ are linearly independent operators acting on $\mathcal{M}$. we suppose that $\sum_{1\le i\le\nu}a_iK_i(\lambda)$ acts trivially on $\mathcal{M}$ for some $a_i\in\mathbf{C}$. Then

$$0=\sum_{1\le i\le\nu}a_iK_i(\lambda).(e^{\lambda+d_j}\otimes 1)=a_j.$$

for $j=1.\ldots.\nu$. This implies that $K_1(\lambda).K_2(\lambda).\ldots.K_\nu(\lambda)$ are linearly independent operators on $\mathcal{M}$. and $\dim(\mathrm{Ker}(\pi_\lambda))\ge\nu$.

To finish the proof of (i). we recall that $\tilde{\mathcal{G}}[\theta]$ is spanned by the following operators of the form

$$x_{c_0}(n.\ \alpha_1+\sum_{1\le i\le\nu}m_ic_i).\quad x_{\alpha_1}(2n+1.\ \sum_{1\le i\le\nu}m_ic_i).$$

$$x_\beta(2n, \sum_{1\le i\le\nu} m_ic_i),$$

for $\beta \in \mathbf{Z}c_0 + \sum_{1\le i\le\nu}(\mathbf{Z}c_i + \mathbf{Z}d_i)$, and $n, m_i \in \mathbf{Z}$, $1 \le i \le \nu$.

We also recall that $x_\beta(m, \alpha)$ acts on $\mathcal{M}$ with degree $(-m, \sum_{1\le i\le\nu}(\alpha, d_i)c_i)$, for $\alpha \in \mathbf{Z}\alpha_1 + \sum_{1\le i\le\nu}\mathbf{Z}c_i$, $\beta \in \Gamma$. Thus, the kernel $\mathrm{Ker}(\pi_\lambda)$ is spanned by its homogeneous elements.

Note that the operators $x_{c_0}(n, \alpha_1)$, and $x_{\alpha_1}(2n+1, 0)$ $(= \alpha_1(2n+1))$, $x_{c_0}(0,0)$ $(= c_0)$ form an affine Lie algebra $A_1^{(1)}$, which is faithfully representated on the subspace $e^\lambda \otimes S(\sum_{k<0}\mathbf{C}\alpha_1(2k+1)) \in \mathcal{M}(\lambda)$.

For $m, n \in \mathbf{Z}$, the operators $x_{c_0}(2n+1, \alpha_1+\sum_{1\le i\le\nu} m_ic_i)$, $x_{\alpha_1}(2n+1, \sum_{1\le i\le\nu} m_ic_i)$, act on $\mathcal{M}$ with the same degree, and $x_{c_0}(2n, \sum_{1\le i\le\nu} m_ic_i)$, $x_{c_0}(2n, \alpha_1+\sum_{1\le i\le\nu} m_ic_i)$, $x_{c_j}(2n, \sum_{1\le i\le\nu} m_ic_i)$, $x_{d_j}(2n, \sum_{1\le i\le\nu} m_ic_i)$, $(1 \le j \le \nu)$, act on $\mathcal{M}$ with the same degree.

Claim Let $n, m_i \in \mathbf{Z}$, $1 \le i \le \nu$.

(a). If $Ax_{c_0}(2n+1, \alpha_1 + \sum_{1\le i\le\nu} m_ic_i) + Bx_{\alpha_1}(2n+1, \sum_{1\le i\le\nu} m_ic_i) \in \mathrm{Ker}(\pi_\lambda)$, for some $A, B \in \mathbf{C}$, then $A = B = 0$.

(b).Let

$$\tau := Ax_{c_0}(2n, \sum_{1\le i\le\nu} m_ic_i) + Bx_{c_0}(2n, \alpha_1 + \sum_{1\le i\le\nu} m_ic_i) + \sum_{j=1}^{\nu}\{A_jx_{c_j}(2n, \sum_{1\le i\le\nu} m_ic_i)$$

$$+B_jx_{d_j}(2n, \sum_{1\le i\le\nu} m_ic_i)\},$$

for some $A, B, A_j, B_j \in \mathbf{C}$. If $\tau \in \mathrm{Ker}(\pi_\lambda)$, then $B = B_j = 0$, for $1 \le j \le \nu$.

Proof of Claim. (a). By Lemma 1.5.4 and Lemma 1.5.6, we have

$$[Ax_{c_0}(2n+1, \alpha_1 + \sum_{1\le i\le\nu} m_ic_i) + Bx_{\alpha_1}(2n+1, \sum_{1\le i\le\nu} m_ic_i), x_{c_0}(-2n, \alpha_1 - \sum_{1\le i\le\nu} m_ic_i)]$$

$$= 2Ax_{\alpha_1}(1, 0) + 2Bx_{c_0}(1, \alpha_1).$$

This belongs to the affine Lie algebra $A_1^{(1)}$, thus is faithfully representaed on the subspace $e^\lambda \otimes S(\sum_{k<0}\alpha_1(2k+1)$ of $\mathcal{M}(\lambda)$, so $A = B = 0$, as required.

(b). For any $k_i \in \mathbf{Z} \setminus \{0\}$, $1 \le i \le \nu$, we have, by Lemma 1.6.1

$$[[\tau, x_{c_0}(-2n, \sum_{1\le i\le\nu}(k_i - m_i)c_i)], x_{d_j}(0, -\sum_{1\le i\le\nu} k_i c_i)]$$

$$= \sum_{l=1}^{\nu} B_l(k_l - m_l)[x_{c_0}(0, \sum_{1\le i\le\nu} k_i c_i), x_{d_j}(0, -\sum_{1\le i\le\nu} k_i c_i)]$$

$$= -\sum_{l=1}^{\nu} B_l(k_l - m_l)k_j x_{c_0}(0, 0),$$

for $1 \le j \le \nu$.

This gives that $\sum_{l=1}^{\nu} B_l(k_l - m_l) = 0$ for all $k_l \in \mathbf{Z} \setminus \{0\}$. Therefore, $B_l = 0$ for $l = 1, \dots, \nu$.

To see $B = 0$, we compute, by Lemma 1.5.6

$$[\tau, x_{c_0}(1 - 2n, \alpha_1 - \sum_{1\le i\le\nu} m_i c_i)]$$

$$= B[x_{c_0}(2n, \alpha_1 + \sum_{1\le i\le\nu} m_i c_i), x_{c_0}(1 - 2n, \alpha_1 - \sum_{1\le i\le\nu} m_i c_i)]$$

$$= -2Bx_{\alpha_1}(1, 0).$$

This gives that $B = 0$. Thus we complete the proof of the claim.

Now we prove (i). By the Claim, we know that the homogeneous elements of $\mathrm{Ker}(\pi_\lambda)$ have the form

$$\tau := Ax_{c_0}(2n, \sum_{1\le i\le\nu} m_i c_i) + \sum_{j=1}^{\nu} A_j x_{c_j}(2n, \sum_{1\le i\le\nu} m_i c_i), \tag{1.7.11}$$

for some $A, A_j \in \mathbf{C}$, and $n, m_i \in \mathbf{Z}$.

Suppose $\tau \neq 0$, and $m_l \neq 0$ for some $1 \le l \le \nu$. Note that $x_{c_0}(2k, 0) = \delta_{k,0} c_0$, and $x_{c_j}(2k, 0) = c_j(2k)$ for $k \in \mathbf{Z}$. We have, by Lemma 1.6.1 and Lemma 1.6.2

$$[x_{d_l}(2k - 2n, -\sum_{1\le i\le\nu} m_i c_i), \tau] \tag{1.7.12}$$

$$= Am_l x_{c_0}(2k, 0) + \sum_{j=1}^{\nu} A_j\{m_l x_{c_j}(2k, 0) + \delta_{lj}\sum_{p=1}^{\nu}(-m_p)x_{c_p}(2k, 0)$$

$$+\delta_{lj}(2k - 2n)x_{c_0}(2k, 0)\}$$

$$= (Am_l + A_l(2k - 2n))x_{c_0}(2k, 0) + \sum_{j=1}^{\nu}(A_j m_l - A_l m_j)x_{c_j}(2k, 0)$$

$$= (Am_l + A_l(2k - 2n))\delta_{k,0}c_0 + \sum_{j=1}^{\nu}(A_j m_l - A_l m_j)c_j(2k).$$

Thus, we have, for $s = 1, \ldots, \nu$, and $k \in \mathbf{Z}_+$,

$$0 = [x_{d_l}(2k - 2n, -\sum_{1\leq i\leq\nu} m_i c_i),\ \tau].e^{\lambda} \otimes d_s(-2k)$$

$$= \sum_{j=1}^{\nu}(A_j m_l - A_l m_j)c_j(2k).e^{\lambda} \otimes d_s(-2k)$$

$$= \{\sum_{j=1}^{\nu}(A_j m_l - A_l m_j)2k\delta_{j,s}\frac{1}{2}\}e^{\lambda} \otimes 1$$

$$= k(A_s m_l - A_l m_s)e^{\lambda} \otimes 1,$$

which implies that $A_s m_l - A_l m_s = 0$, for all $s = 1, \ldots, \nu$, that is, (since we assumed $m_l \neq 0$)

$$A_s = \frac{A_l}{m_l}m_s, \tag{1.7.13}$$

for all $s = 1, \ldots, \nu$.

Moreover, from this and (1.7.12), we also obtain

$$0 = [x_{d_l}(-2n, -\sum_{1\leq i\leq\nu} m_i c_i,\ \tau].e^{\lambda} \otimes 1$$

$$= (Am_l + A_l(-2n))c_0.e^{\lambda} \otimes 1 = \frac{1}{2}(Am_l - 2nA_l)e^{\lambda} \otimes 1.$$

This gives that $Am_l - 2nA_l = 0$, or

$$A = \frac{A_l}{m_l}(2n). \tag{1.7.14}$$

We substitute (1.7.13)(1.7.14) into (1.7.11), and obtain

$$\tau = \frac{A_l}{m_l}2nx_{c_0}(2n, \sum_{1\leq i\leq\nu} m_i c_i) + \sum_{j=1}^{\nu}\frac{A_l}{m_l}m_j x_{c_j}(2n, \sum_{1\leq i\leq\nu} m_i c_i)$$

$$= \frac{A_l}{m_l}\{2nx_{c_0}(2n, \sum_{1\leq i\leq\nu} m_i c_i) + \sum_{j=1}^{\nu} m_j x_{c_j}(2n, \sum_{1\leq i\leq\nu} m_i c_i)\}.$$

This is zero by Lemma 1.5.2, and contradicts the assumption $\tau \neq 0$. Therefore, $m_l = 0$ for all $1 \leq l \leq \nu$, and, if $\tau \in \mathrm{Ker}(\pi_\lambda)$, $\tau \neq 0$, then τ must be of the following form

$$\tau = Ax_{c_0}(2n,0) + \sum_{j=1}^{\nu} A_j x_{c_j}(2n,0),$$

for some $n \in \mathbf{Z}$, $A, A_j \in \mathbf{C}$, that is,

$$\tau = A\delta_{n,0}c_0 + \sum_{j=1}^{\nu} A_j c_j(2n). \tag{1.7.15}$$

We claim that, in (1.7.15), n must be zero. Otherwise, suppose that $n < 0$, then

$$0 = \tau.e^\lambda \otimes 1 = \sum_{j=1}^{\nu} A_j c_j(2n).e^\lambda \otimes 1 = \sum_{j=1}^{\nu} A_j.e^\lambda \otimes c_j(2n).$$

This gives $A_j = 0$, for $1 \leq j \leq \nu$, since $\{e^\lambda \otimes c_j(2n)\}_{j=1,\dots,\nu}$ are linearly independent elements of $M(\lambda)$. Thus, $\tau = 0$, which is a contradiction.

Similarly, if $n > 0$, then we have, for $1 \leq l \leq \nu$,

$$0 = \tau.e^\lambda \otimes d_l(-2n) = nA_l e^\lambda \otimes 1.$$

This also gives $\tau = 0$, and a contradiction.

Therefore, if $\tau \in \mathrm{Ker}(M(\lambda))$ and $\tau \neq 0$, then

$$\tau = Ac_0 + \sum_{j=1}^{\nu} A_j c_j(0), \tag{1.7.16}$$

for some $A, A_j \in \mathbf{C}$.

But,

$$0 = \tau.e^\lambda \otimes 1 = (A/2 + \sum_{j=1}^{\nu} A_j(\lambda, c_j))e^\lambda \otimes 1,$$

so,

$$A/2 + \sum_{j=1}^{\nu} A_j(\lambda, c_j) = 0,$$

that is

$$A = -2\sum_{j=1}^{\nu} A_j(\lambda, c_j).$$

Hence, from this and (1.7.16), we have

$$\tau = -2\sum_{j=1}^{\nu} A_j(\lambda, c_j)c_0 + \sum_{j=1}^{\nu} A_j c_j(0)$$

$$= \sum_{j=1}^{\nu} A_j(c_j(0) - 2(\lambda, c_j)c_0) = \sum_{j=1}^{\nu} A_j K_j(\lambda).$$

This completes the proof of (i).

Finally, (ii) follows from the fact that $K_1(\lambda), \ldots, K_\nu(\lambda)$ are linearly independent operators on $\mathcal{M}$, and what we learned from the result (i).

$\square$

§1.8 Second Construction of $\hat{\mathcal{G}}[\theta]$ by Vertex Operators

In this section, we study another representation of the toroidal Lie algebra $\hat{\mathcal{G}}[\theta]$. The construction is a generalization of the so-called level two representation of the affine Lie algebra $A_1^{(1)}$ by (principal) vertex operators. Such a construction of $A_1^{(1)}$ was first obtained by Lepowsky and Wilson[LW2]. Here we extend this construction to the toridal case.

Let $\mathcal{W}$ be a Clifford algebra with generators ω_i for $i \in 2\mathbf{Z}+1$, and relation

$$\omega_i\omega_j + \omega_j\omega_i = -2\delta_{i+j,0}, \tag{1.8.1}$$

for $i, j \in 2\mathbf{Z}+1$. Let $\Lambda(\mathcal{W}^-)$ be the standard irreducibal representation, i.e the exterior algebra on the generators ω_{-j} with $j \in 2\mathbf{N}+1$.The action of $\mathcal{W}$ on $\Lambda(\mathcal{W}^-)$ is determined by

$$\omega_j.1 = 0, \quad \omega_{-j}.\omega = \omega_{-j}\wedge\omega, \tag{1.8.2}$$

and

$$\omega_i.(\omega_{-j}\wedge\omega) = -2\delta_{i-j,0}\omega - \omega_{-j}\wedge(\omega_i.\omega), \tag{1.8.3}$$

for $i, j \in 2\mathbf{N}+1$, $\omega \in \Lambda(\mathcal{W}^-)$.

Let $\Gamma_1 = \sum_{1\le i\le \nu}(\mathbf{Z}c_i + 2\mathbf{Z}d_i) \subset \Gamma_0$. Then $\mathcal{M}_1 := \mathbf{C}[\Gamma_1] \otimes \mathcal{S}(\hat{\mathcal{H}}^-)$ is an $\hat{\mathcal{H}}$-submodule of $\mathcal{M}$. Now we extand the action of $\hat{\mathcal{H}}$ on $\mathcal{M}_1$ to a representation of $\hat{\mathcal{H}}$ on the following vector space

$$\mathcal{M}_1(\mathcal{W}) := \mathcal{M}_1 \otimes \Lambda(\mathcal{W}^-). \tag{1.8.4}$$

by defining $c_0 \mapsto 1$, $\alpha(m) \mapsto \alpha(m) \otimes 1$, for $m \in \mathbf{Z}$, $\alpha \in H$.

We define the action of $\mathcal{W}$ on $\mathcal{M}_1(\mathcal{W})$ by requiring ω_i acts as $1 \otimes \omega_i$, for $i \in 2\mathbf{Z}+1$. For convenience, we set

$$\omega_{2i} = \delta_{i,0}, \quad \text{for } \ i \in \mathbf{Z}, \tag{1.8.5}$$

and define

$$W^{\alpha}(z) = \sum_{j\in 2\mathbf{Z}} \omega_{j+\frac{(\alpha,\alpha)}{2}} z^{-j-\frac{(\alpha,\alpha)}{2}}, \tag{1.8.6}$$

for $\alpha \in \mathbf{Z}\alpha_1 + \sum_{1\le i\le \nu}\mathbf{Z}c_i$, and

$$W(z) = \sum_{j\in 2\mathbf{Z}+1} \omega_j z^{-j}. \tag{1.8.7}$$

It is clear that $W^{\alpha_1}(z) = W(z)$. The following Lemma comes from [L1], and hence we only indicate the proof.

Lemma 1.8.1 We have

$$\begin{aligned} W(z_1)W(z_2)(1-\frac{z_2}{z_1})(1+\frac{z_2}{z_1})^{-1} - W(z_2)W(z_1)(1-\frac{z_1}{z_2})(1+\frac{z_1}{z_2})^{-1} \\ = 2(D\delta)(-\frac{z_2}{z_1}). \end{aligned} \tag{1.8.8}$$

Proof. We have

$$A := W(z_1)W(z_2)(1-\frac{z_2}{z_1})(1+\frac{z_2}{z_1})^{-1} - W(z_2)W(z_1)(1-\frac{z_1}{z_2})(1+\frac{z_1}{z_2})^{-1}$$

$$= \sum_{i,j\in 2\mathbf{Z}+1} \omega_i\omega_j z_1^{-i}z_2^{-j}(1+2\sum_{k=1}^{\infty}(-1)^k z_1^{-k}z_2^{k}) - \sum_{i,j\in 2\mathbf{Z}+1} \omega_j\omega_i z_1^{-i}z_2^{-j}(1+2\sum_{k=1}^{\infty}(-1)^k z_1^{k}z_2^{-k})$$

$$= \sum_{\substack{i,j\in\mathbf{Z}\\ i+j\in 2\mathbf{Z}}} b_{ij} z_1^{-i}z_2^{-j}.$$

where an easy computation shows that, for $i, j \in 2\mathbf{Z}$

$$b_{ij} = \begin{cases} -2(\omega_{i-1}\omega_{j+1} + \omega_{i-3}\omega_{j+3} + \cdots + \omega_{j+1}\omega_{i-1}), & \text{if } i > j, \\ 0, & \text{if } i = j, \\ 2(\omega_{j-1}\omega_{i+1} + \omega_{j-3}\omega_{i+3} + \cdots + \omega_{i+1}\omega_{j-1}), & \text{if } i < j, \end{cases}$$

and, for $i, j \in 2\mathbf{Z} + 1$,

$$b_{ij} = \begin{cases} \omega_i\omega_j + 2(\omega_{i-2}\omega_{j+2} + \omega_{i-4}\omega_{j+4} + \cdots + \omega_{j+2}\omega_{i-2}) + \omega_j\omega_i, & \text{if } i > j, \\ 0, & \text{if } i = j, \\ -\omega_j\omega_i - 2(\omega_{j-2}\omega_{i+2} + \omega_{j-4}\omega_{i+4} + \cdots + \omega_{i+2}\omega_{j-2}) - \omega_i\omega_j, & \text{if } i < j. \end{cases}$$

But in all cases, one can see that

$$b_{ij} = -2i(-1)^{i+1}\delta_{i+j,0}.$$

Therefore

$$A = \sum_{i,j\in\mathbf{Z}} (-2i(-1)^{i+1}\delta_{i+j,0}) z_1^{-i} z_2^{-j} = 2\sum_{i\in\mathbf{Z}} i(-\frac{z_2}{z_1})^i = 2(D\delta)(-\frac{z_2}{z_1}),$$

as required.

$\square$

In what follows, we want to extend the representation of $\hat{\mathcal{H}}$ on $\mathcal{M}_1(\mathcal{W})$ to a representation of the toroidal Lie algebra $\hat{\mathcal{G}}[\theta]$, as well as to its extension $\check{\mathcal{G}}[\theta]$, by vertex opertors on $\mathcal{M}_1(\mathcal{W})$.

We define the vertex operator by

$$Y_\beta(\alpha, z) = \epsilon^\alpha z^\alpha F^-(\alpha, -z)\beta(-z)W^\alpha(z)F^+(\alpha, -z), \tag{1.8.9}$$

for $\alpha \in \mathbf{Z}\alpha_1 + \sum_{1\le j\le\nu}\mathbf{Z}c_j$, $\beta \in \mathbf{Z}\alpha_1 + \sum_{1\le j\le\nu}(\mathbf{Z}c_j + \mathbf{Z}d_j) + \mathbf{Z}c_0$, where

$$F^\pm(\alpha, z) = \exp(-\sum_{j\in\mathbf{Z}_\pm} \frac{\alpha(j)}{j} z^{-j}),$$

and the operators $\epsilon^\alpha, z^\alpha$ are understood as $\epsilon^\alpha \otimes 1, z^\alpha \otimes 1$ respectively.

We define the operators $y_\beta(j, \alpha)$ $(j \in \mathbf{Z})$ on $\mathcal{M}_1(\mathcal{W})$ by formally expanding the vertex operator

$$Y_\beta(\alpha, z) = \sum_{j\in\mathbf{Z}} y_\beta(j, \alpha) z^{-j}. \tag{1.8.10}$$

We have the following two results

Theorem 1.8.2 The Lie algebra of operators, acting on $\mathcal{M}_1(\mathcal{W})$, spanned by the operators

$$\{y_{\alpha_1}(2n+1, \sum_{1\le i\le \nu} m_i c_i), \quad y_{c_0}(n, \alpha_1 + \sum_{1\le i\le \nu} m_i c_i), \tag{1.8.11}$$
$$y_{c_0}(2n, \sum_{1\le i\le \nu} m_i c_i), \quad y_{c_j}(2n, \sum_{1\le i\le \nu} m_i c_i)\},$$

for $j = 1, \ldots, \nu$, is isomorphic to the toroidal Lie algebra $\hat{\mathcal{G}}[\theta]$ by the map ι':

$$y_{\alpha_1}(2n+1, \sum_{1\le i\le \nu} m_i c_i) \mapsto \alpha_1(2n+1, \vec{m}), \tag{1.8.12}$$
$$y_{c_0}(n, \alpha_1 + \sum_{1\le i\le \nu} m_i c_i) \mapsto X(n, \vec{m}),$$
$$y_{c_0}(2n, \sum_{1\le i\le \nu} m_i c_i) \mapsto C_0(2n, \vec{m}),$$
$$y_{c_j}(2n, \sum_{1\le i\le \nu} m_i c_i) \mapsto C_j(2n, \vec{m}),$$

where $\vec{m} = (m_1, \ldots, m_\nu) \in \mathbf{Z}^\nu$, $j = 1, \ldots, \nu$, $n \in \mathbf{Z}$.

Theorem 1.8.3 The Lie algebra of operators of $\mathcal{M}_1(\mathcal{W})$, spanned by the operators $y_{d_j}(2n, \sum_{1\le i\le \nu} m_i c_i)$, $j = 1, \ldots, \nu$, and the operators given in Theorem 1.8.2, is isomorphic to the extended toroidal Lie algebra $\hat{\mathcal{G}}[\theta] \dotplus \mathcal{D}$ (see (1.4.10)). The isomorphism is defined by saying it extends the map ι' [see (1.8.12)] and satisfies

$$y_{d_j}(2n, \sum_{1\le i\le \nu} m_i c_i) \mapsto d_j(2n, \vec{m}),$$

for $j = 1, 2, \ldots, \nu$.

Proof of Theorem 1.8.2 and Theorem 1.8.3:

If $\alpha \in \sum_{1\le i\le \nu} \mathbf{Z}c_i$, then $(\alpha, \alpha) = 0$, and $W^\alpha(z) = \sum_{j\in 2\mathbf{Z}} \omega_j z^{-j} = 1$. Thus we have

$$Y_\beta(\alpha, z) = \epsilon^\alpha z^\alpha F^-(\alpha, -z)\beta(-z)F^+(\alpha, -z), \tag{1.8.13}$$

for $\alpha \in \sum_{1\le i\le \nu} \mathbf{Z}c_i$, $\beta \in \mathbf{Z}\alpha_1 + \sum_{1\le i\le \nu}(\mathbf{Z}c_i + \mathbf{Z}d_i) + \mathbf{Z}c_0$.

We note that on the right hand side of (1.8.13), there is no operator which arises from the representation of the Clifford algebra $\mathcal{W}$. Thus $Y_\beta(\alpha, z)$, just as $X_\beta(\alpha, z)$ from our first construction, acts only on the first component of the Fock space $\mathcal{M}_1(\mathcal{W}) = \mathcal{M}_1 \otimes \Lambda(\mathcal{W}^-)$. Therefore to check the commutation relations corresponding to (1.2.21)(1.2.23) and (1.6.1)- (1.6.5), one may use the same arguments as we did in the proof of Lemma 1.5.4- 1.5.5 and Lemma 1.6.1-1.6.3. Hence we omit these similar arguments.

It follows from the above that we must only check the following commutation relation

(1.8.14)
$$[Y_{c_0}(\alpha_1 + \sum_{1\le i\le \nu} m_i c_i, z), Y_{c_0}(\alpha_1 + \sum_{1\le i\le \nu} n_i c_i, w)]$$
$$= 2Y_{c_0}(\sum_{1\le i\le \nu}(m_i+n_i)c_i, w)(D\delta)(-\frac{w}{z}) - 2\{Y_{\alpha_1}(\sum_{1\le i\le \nu}(m_i+n_1)c_i, w)$$
$$-\sum_{j=1}^{\nu} m_j Y_{c_j}(\sum_{1\le i\le \nu}(m_i+n_i)c_i, w)\}\delta(-\frac{w}{z}).$$

for $(m_1, \ldots, m_\nu)$, $(n_1, \ldots, n_\nu) \in \mathbf{Z}^\nu$. This corresponds to the relation (1.2.23). Indeed, this is the only one involving the computation of the operators arising from the Clifford algebra $\mathcal{W}$.

To check (1.8.14), we set $\alpha = \alpha_1 + \sum_{1\le i\le \nu} m_i c_i$, $\beta = \alpha_1 + \sum_{1\le i\le \nu} n_i c_i$. Note that $c_0(-z) = c_0(-w) = 1$, and $W^\alpha(z) = W(z)$, $W^\beta(w) = W(w)$ as $(\alpha, \alpha) = (\beta, \beta) = 2$. We compute

(1.8.15)
$$Y_{c_0}(\alpha, z)Y_{c_0}(\beta, w)$$
$$= \epsilon^\alpha z^\alpha F^-(\alpha, -z)W(z)F^+(\alpha, -z)\cdot \epsilon^\beta w^\beta F^-(\beta, -w)W(w)F^+(\beta, -w)$$
$$= \epsilon^{\alpha+\beta} z^\alpha w^\beta F^-(\alpha, -z)W(z)F^+(\alpha, -z)F^-(\beta, -w)W(w)F^+(\beta, -w),$$

where, as (1.5.5)
$$F^+(\alpha, -z)F^-(\beta, -w)$$
$$= F^-(\beta, -w)F^+(\alpha, -z)\exp\{[-\sum_{j\in\mathbf{Z}_+}\frac{\alpha(j)}{j}(-z)^{-j}, -\sum_{j\in\mathbf{Z}_-}\frac{\beta(j)}{j}(-w)^{-j}]\}$$

$$= F^-(\beta, -w)F^+(\alpha, -z)\exp\{-\sum_{j,l\in 2\mathbf{N}+1}[\alpha_1(j), \alpha_1(-l)]z^{-j}w^l/(jl)\}$$

$$= F^-(\beta, -w)F^+(\alpha, -z)\exp(-2\sum_{l\in 2\mathbf{Z}+1} l^{-1}(\frac{w}{z})^l)$$

$$= F^-(\beta, -w)F^+(\alpha, -z)(1-\frac{w}{z})(1+\frac{w}{z})^{-1}.$$

in the last equality, we have used the fact (see [FLM], P76)

$$-2\sum_{l\in 2\mathbf{Z}+1} l^{-1}x^l = \log[(1-x)(1+x)^{-1}].$$

and the standard rules

$$\exp(\log(1+ax)) = 1+ax, \quad \log((1+ax)(1+bx)) = \log(1+ax)+\log(1+bx),$$

$$\log(1+ax)^b = b\log(1+ax),$$

for $a, b \in \mathbf{C}^\times$.

Therefore, we obtain, from (1.8.15)

(1.8.16)
$$Y_{c_0}(\alpha, z)Y_{c_0}(\beta, w)$$
$$= \epsilon^{\alpha+\beta}z^\alpha w^\beta F^-(\alpha, -z)F^-(\beta, -w)(1-\frac{w}{z})(1+\frac{w}{z})^{-1}W(z)W(w)$$
$$\cdot F^+(\alpha, -z)F^+(\beta, -w).$$

Similarly,

$$Y_{c_0}(\beta, w)Y_{c_0}(\alpha, z)$$
$$= \epsilon^{\alpha+\beta}z^\alpha w^\beta F^-(\alpha, -z)F^-(\beta, -w)(1-\frac{z}{w})(1+\frac{z}{w})^{-1}W(w)W(z)$$
$$\cdot F^+(\alpha, -z)F^+(\beta, -w).$$

Thus, the above two identities give us

(1.8.17)
$$[Y_{c_0}(\alpha, z), Y_{c_0}(\beta, w)]$$
$$= \epsilon^{\alpha+\beta}z^\alpha w^\beta F^-(\alpha, -z)F^-(\beta, -w)\{(1-\frac{w}{z})(1+\frac{w}{z})^{-1}W(z)W(w)$$
$$-(1-\frac{z}{w})(1+\frac{z}{w})^{-1}W(w)W(z)\}F^+(\alpha, -z)F^+(\beta, -w)$$

$$= e^{\alpha+\beta} z^{\alpha} w^{\beta} F^{-}(\alpha, -z) F^{-}(\beta, -w) F^{+}(\alpha, -z) F^{+}(\beta, -w)\{2(D\delta)(-\frac{w}{z})\},$$

where we have used Lemma 1.8.1.

Notice that, as operators on $\mathcal{M}_1(\mathcal{W})$, $(-z)^{\gamma} = z^{\gamma}$, for $\gamma \in \mathbf{Z}\alpha_1 + \sum_{1\le i\le \nu} \mathbf{Z}c_i$, and $e^{\alpha_1+\sum_{1\le i\le\nu} m_i c_i} = e^{\sum_{1\le i\le\nu} m_i c_i}$, $z^{\alpha_1+\sum_{1\le i\le\nu} m_i c_i} = z^{\sum_{1\le i\le\nu} m_i c_i}$, for $m_i \in \mathbf{Z}$ [see (1.4.4)]. Moreover, we have

$$\begin{aligned}
&F^{\pm}(\alpha, w) F^{\pm}(\beta, -w)\\
&= \exp(-\sum_{j\in\mathbf{Z}_{\pm}} \frac{\alpha(j)}{j} w^{-j}) \exp(-\sum_{j\in\mathbf{Z}_{\pm}} \frac{\beta(j)}{j}(-w)^{-j})\\
&= \exp\{-\sum_{j\in\mathbf{Z}_{\pm}} \frac{w^{-j}}{j}(\alpha(j) + (-1)^{j}\beta(j))\}\\
&= \exp\{-\sum_{j\in 2\mathbf{Z}_{\pm}} \frac{w^{-j}}{j} \sum_{i=1}^{\nu}(m_i + n_i)c_i(j)\} = F^{\pm}(\sum_{i=1}^{\nu}(m_i + n_i)c_i, -w).
\end{aligned}$$

Therefore, applying Lemma 1.5.1, we obtain from (1.8.17)

$$\begin{aligned}
&[Y_{c_0}(\alpha, z), Y_{c_0}(\beta, w)]\\
&= 2e^{\sum_{i=1}^{\nu}(m_i+n_i)c_i} w^{\sum_{i=1}^{\nu}(m_i+n_i)c_i} F^{-}(\sum_{i=1}^{\nu}(m_i + n_i)c_i, -w)\\
&\quad \cdot F^{+}(\sum_{i=1}^{\nu}(m_i + n_i)c_i, -w)(D\delta)(-\frac{w}{z})\\
&\quad +2e^{\sum_{i=1}^{\nu}(m_i+n_i)c_i} w^{\sum_{i=1}^{\nu}(m_i+n_i)c_i} F^{-}(\sum_{i=1}^{\nu}(m_i + n_i)c_i, -w)\{\alpha(0)\\
&\quad + \sum_{j\in\mathbf{Z}_+} \alpha(j)w^{-j} + \sum_{j\in\mathbf{Z}_-} \alpha(j)w^{-j}\} F^{+}(\sum_{i=1}^{\nu}(m_i + n_i)c_i, -w)\delta(-\frac{w}{z})\\
&= 2Y_{c_0}(\sum_{i=1}^{\nu}(m_i + n_i)c_i, w)(D\delta)(-\frac{w}{z}) + 2\{-Y_{\alpha_1}(\sum_{i=1}^{\nu}(m_i + n_i)c_i, w)\\
&\quad + \sum_{j=1}^{\nu} m_j Y_{c_j}(\sum_{i=1}^{\nu}(m_i + n_i)c_i, w)\}\delta(-\frac{w}{z}),
\end{aligned}$$

as required.

□

Chapter 2

TKK Algebra and Its Universal Central Extension

§2.1 Introduction

The TKK algebra $\mathcal{K}(\mathcal{J})$ is obtained from a Jordan algebra $\mathcal{J}$ by using the Tits-Kantor-Koecher construction (see[J1, VIII.5]). In [AABGP], they start with a semi-lattice S to define a Jordan algebra $\mathcal{J} = \mathcal{J}(S)$. From $\mathcal{J}(S)$ they contruct an example of an extended affine Lie algebra with an extended affine root system of type A_1.

When one specializes this construction to the smallest possible semi-lattice in two variables, one obtains the smallest extended affine Lie algebra which is not of finite or affine type. That is, it has the smallest possible root system which is not finite or affine (see [AABGP]). For this reason we will call the extended affine Lie algebra arising from this the Baby algebra. We study this Baby algebra extensively in this and the next chapter.

We will see (in Proposition 2.2.3) that $\mathcal{K}(\mathcal{J})$ is isomorphic to the following Lie algebra

$$\mathcal{G}(\mathcal{J}) := sl_2(\mathbb{C}) \otimes \mathcal{J} \oplus \mathrm{Inder}(\mathcal{J}).$$

where $\mathrm{Inder}(\mathcal{J})$ is the set of inner derivations of the Jordan algebra $\mathcal{J}$. Thus this TKK algebra $\mathcal{K}(\mathcal{J})$ comes from the smallest finite dimensional simple Lie algebra $sl_2(\mathbb{C})$. We will indeed concerntrate on a Jordan algebra $\mathcal{J} := \mathcal{J}(S)$, where S is the 'smallest' (non-lattice) semilattice, and this will be the coordinate of the Baby algebra.

This chapter is organized as follows. In next section we will recall the terminology and notation that we need from [AABGP]. We will identify the TKK algebra $\mathcal{K}(\mathcal{J})$ with $\mathcal{G}(\mathcal{J})$ via an isomorphism φ. In Section 2.3 we discuss the universal central extension $\hat{\mathcal{G}}(\mathcal{J})$ of $\mathcal{G}(\mathcal{J})$. Section 2.4 is devoted to the study of the centre $\mathcal{Z}(\hat{\mathcal{G}}(\mathcal{J}))$ of $\hat{\mathcal{G}}(\mathcal{J})$, which in fact is given by a Jordan algebra version of the Connes cyclic homology group $HC_1(\mathcal{J})$. We will just call $HC_1(\mathcal{J})$ the Connes cyclic homology group of $\mathcal{J}$. Finally, in Section 2.5 we describe the structure of $\hat{\mathcal{G}}(\mathcal{J})$ in terms of identities of formal power series in formal variables. This will allow us to go on, in the next chapter, to study the vertex operator representation of $\hat{\mathcal{G}}(\mathcal{J})$.

§2.2 TKK Algebra

Let $\nu \geq 1$ be an integer. An additive subgroup Λ of $\mathbf{R}^\nu$ is called a lattice if Λ is discrete and spans $\mathbf{R}^\nu$. A subset S of $\mathbf{R}^\nu$ is called a semilattice of $\mathbf{R}^\nu$ if S satisfies the axioms

(S1). $0 \in S$. $-S = S$. $S + 2S \subseteq S$.

(S2). S spans $\mathbf{R}^\nu$.

(S3). S is discrete in $\mathbf{R}^\nu$.

We note that if S is an additive subgroup of $\mathbf{R}^\nu$, then S satisfies the axiom (S1). Thus a lattice in $\mathbf{R}^\nu$ is a semilattice.

Suppose $S = \cup_{i=0}^m S_i$, where S_i are distinct cosets of $2\mathbf{Z}^\nu$ in $\mathbf{Z}^\nu$, and $S_0 = 2\mathbf{Z}^\nu$. Then S forms a semilattice of $\mathbf{R}^\nu$. For $\sigma \in S$, let x^σ be a symbol. We form a vector space

$$\mathcal{T} = \mathcal{T}(S) = \sum_{\sigma \in S} \dot{+} \mathbf{C} x^\sigma$$

with the multiplication defined by

$$x^\sigma x^\tau = \begin{cases} x^{\sigma+\tau}, & \text{if } \sigma, \tau \in S_0 \cup S_i, \text{ for some } i, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ forms a Jordan algebra with identity $x^0 = 1$, (see [AABGP]). Moreover, we denote by Inder($\mathcal{T}$) the inner derivations of $\mathcal{T}$, then

$$\text{Inder}(\mathcal{T}) = \{\sum_i [L_{b_i}, L_{c_i}] |\, b_i, c_i \in \mathcal{T}\}.$$

Let Instrl($\mathcal{T}$) $= L_{\mathcal{T}} \dot{+}$ Inder($\mathcal{T}$), with the multilication

$$(2.2.1) \qquad [L_a + D, L_b + E] = [L_a, L_b] + L_{Db} - L_{Ea} + [D, E],$$

for $a, b \in \mathcal{T}$, $D, E \in$ Inder($\mathcal{T}$), where the operator $L_a \in L_{\mathcal{T}}$ is defined by $L_a b = ab$ for $a, b \in \mathcal{T}$. Instrl($\mathcal{T}$) is a subalgebra of the Lie algebra $gl(\mathcal{T})$, and is called the inner structure Lie algebra of $\mathcal{T}$.

Instrl($\mathcal{T}$) has an automorphism '$\bar{\ }$' of order two defined by $\overline{L_a + D} = -L_a + D$. We denote by $\bar{\mathcal{T}}$ a linear copy of $\mathcal{T}$, which has element of the form $\bar{a}$ for $a \in \mathcal{T}$. The

Tits-Kantor-Koecher construction (see [Ja]) gives the so called TKK algebra of the form

$$\mathcal{K}(\mathcal{T}) = \mathcal{T} \dotplus \mathrm{Instrl}(\mathcal{T}) \dotplus \bar{\mathcal{T}},$$

with the Lie multilication

(2.2.2) $$[a_1 + \bar{b}_1 + E_1, a_2 + \bar{b}_2 + E_2]$$

$$= -E_2 a_1 + E_1 a_2 - \overline{\bar{E}_2 b_1} + \overline{\bar{E}_1 b_2} + a_1 \triangle b_2 - a_2 \triangle b_1 + [E_1, E_2],$$

for $a_i, b_i \in \mathcal{T}$, $E_i \in \mathrm{Instrl}(\mathcal{T})$, where $a \triangle b = L_{ab} + [L_a, L_b]$.

Lemma 2.2.1 Let $a_i \in \mathcal{T}$, $i = 1, 2, 3$. We have

$$[L_{a_1}, L_{a_2}] + [L_{a_2}, L_{a_1}] = 0,$$

$$[L_{a_1 a_2}, L_{a_3}] + [L_{a_2 a_3}, L_{a_1}] + [L_{a_3 a_1}, L_{a_2}] = 0.$$

Proof. The first identity is clear. For the second one, we note that $[L_a, L_b]c = a(bc) - b(ac)$, and $L_{a+b} = L_a + L_b$, for $a, b \in \mathcal{T}$. Thus, by (2.2.1),

$$[[L_{a_1}, L_{a_2}], L_{a_3}] = L_{[L_{a_1}, L_{a_2}]a_3}$$

$$= L_{a_1(a_2 a_3)} - L_{a_2(a_1 a_3)}.$$

We adopt the notation

(2.2.3) $$[R_1, R_2, R_3] := [[R_1, R_2], R_3] + [[R_2, R_3], R_1] + [[R_3, R_1], R_2].$$

Let $R_1 = a_1$, $R_2 = \bar{a}_2$ and $R_3 = L_{a_3}$. We have

(2.2.4) $$[[R_1, R_2], R_3] = [[a_1, \bar{a}_2], L_{a_3}]$$

$$= [L_{a_1 a_2} + [L_{a_1}, L_{a_2}], L_{a_3}]$$

$$= [L_{a_1 a_2}, L_{a_3}] + [[L_{a_1}, L_{a_2}], L_{a_3}]$$

$$= [L_{a_1 a_2}, L_{a_3}] + L_{a_1(a_2 a_3)} - L_{a_2(a_1 a_3)}.$$

and

$$(2.2.5) \qquad [[R_2, R_3], R_1] + [[R_3, R_1], R_2]$$

$$= [[\bar{a}_2, L_{a_3}], a_1] + [[L_{a_3}, a_1], \bar{a}_2]$$

$$= [\overline{a_3 a_2}, a_1] + [a_3 a_1, \bar{a}_2]$$

$$= -L_{a_1(a_2a_3)} - [L_{a_1}, L_{a_3a_2}] + L_{a_2(a_3a_1)} + [L_{a_3a_1}, L_{a_2}].$$

Therefore, (2.2.4), (2.2.5) give us

$$[R_1, R_2, R_3] = [L_{a_1a_2}, L_{a_3}] + [L_{a_2a_3}, L_{a_1}] + [L_{a_3a_1}, L_{a_2}] = 0,$$

as required.

□

Lemma 2.2.2 For $D \in \mathrm{Inder}(\mathcal{T})$, $a, b \in \mathcal{T}$, we have

$$D(ab) = (Da)b + a(Db),$$

$$[D, [L_a, L_b]] = [L_{Da}, L_b] + [L_a, L_{Db}].$$

Proof. The first identity is clear. The second one follows from the Jacobi identity and (2.2.1).

□

In this Chapter, we let x_+, x_- and α be the standard Chevally basis of $sl_2(\mathbf{C})$, (note we used e, f, h in Chapter 1). We define

$$\mathcal{G}(\mathcal{T}) := sl_2(\mathbf{C}) \otimes \mathcal{T} \oplus \mathrm{Inder}(\mathcal{T}),$$

with the multilication

$$(2.2.6) \qquad [A \otimes a, B \otimes b] = [A, B] \otimes ab + (A, B)[L_a, L_b],$$

$$[D, A \otimes a] = A \otimes Da,$$

$$[D, [L_a, L_b]] = [L_{Da}, L_b] + [L_a, L_{Db}].$$

for $A, B \in sl_2(\mathbf{C})$, $a, b \in \mathcal{T}$, and $D \in \mathrm{Inder}(\mathcal{T})$, where $(A, B) = 2\mathrm{tr}(AB)$. It is well-known that this makes $\mathcal{G}(\mathcal{T})$ into a Lie algebra.

Proposition 2.2.3 $\mathcal{G}(\mathcal{T})$ is a Lie algebra which is isomorphic to the TKK algebra $\mathcal{K}(\mathcal{T})$ via the linear extension of the map $\varphi : \mathcal{G}(\mathcal{T}) \to \mathcal{K}(\mathcal{T})$ defined by

$$x_+ \otimes a \mapsto \sqrt{2}a, \qquad x_- \otimes a \mapsto \sqrt{2}\bar{a},$$

$$\alpha \otimes a \mapsto 2L_a, \qquad [L_a, L_b] \mapsto [L_a, L_b],$$

for $a, b \in \mathcal{T}$.

Proof. It is clear that φ is a well-defined map from $\mathcal{G}(\mathcal{T})$ to $\mathcal{K}(\mathcal{T})$, and φ is onto and one to one. To show φ defines a Lie algebra homomorphism, one only need to check the corresponding Lie commutation relations. For instance, for $a, b \in \mathcal{T}$,

$$\varphi([x_+ \otimes a, x_- \otimes b]) = \varphi(\alpha \otimes ab + 2[L_a, L_b])$$

$$= 2L_{ab} + 2[L_a, L_b] = 2[a, \bar{b}]$$

$$= [\varphi(x_+ \otimes a), \varphi(x_- \otimes b)].$$

Other commutation relations can be checked similarly.

□

In what follows, we shall identity $\mathcal{G}(\mathcal{T})$ with the TKK algebra $\mathcal{K}(\mathcal{T})$ via this isomorphism.

§2.3 Universal Central Extension of the TKK Algebra

Let $\mathcal{G}$ be a perfect Lie algebra. A central extension of $\mathcal{G}$ is a Lie algebra $\hat{\mathcal{G}}$ and a surjective homomorphism $\pi : \hat{\mathcal{G}} \to \mathcal{G}$ whose kernel lies in the center $\mathcal{Z}(\hat{\mathcal{G}})$ of $\hat{\mathcal{G}}$. The pair $(\hat{\mathcal{G}}, \pi)$ is called a covering central extension of $\mathcal{G}$ if in addition $\hat{\mathcal{G}}$ is perfect. A covering central extension $(\hat{\mathcal{G}}, \pi)$ is a universal central extension of $\mathcal{G}$ if for every central extension (ϵ, φ) of $\mathcal{G}$ there exists a unique homomorphism $\psi : \hat{\mathcal{G}} \to \epsilon$ so that

$\varphi\psi = \pi$. It is well-known [Gar] that every perfect Lie algebra has a universal central extension.

By checking the commutation relations (2.2.6) of the TKK algebra $\mathcal{G}(\mathcal{T})$, we know that $\mathcal{G}(\mathcal{T})$ is a perfect, center free Lie algebra. In order to describe the universal central extension of $\mathcal{G}(\mathcal{T})$, we consider the quotient space

$$< \mathcal{T}, \mathcal{T} > := \mathcal{T} \otimes \mathcal{T}/I,$$

where I is the subspace of $\mathcal{T} \otimes \mathcal{T}$ spanned by the elements

$$a \otimes b + b \otimes a, \qquad ab \otimes c + bc \otimes a + ca \otimes b,$$

for $a, b, c \in \mathcal{T}$. We let $< a, b >$ denote the element $a \otimes b + I$, so we have

$$(2.3.1) \qquad < a, b > + < b, a > = 0, \quad < ab, c > + < bc, a > + < ca, b > = 0,$$

for $a, b, c \in \mathcal{T}$.

Now we define an algebra $\hat{\mathcal{G}}(\mathcal{T})$ by

$$\hat{\mathcal{G}}(\mathcal{T}) := sl_2(\mathbf{C}) \otimes \mathcal{T} + < \mathcal{T}, \mathcal{T} >,$$

with the following multiplications

$$(2.3.2) \qquad [A \otimes a, B \otimes b] = [A, B] \otimes ab + (A, B) < a, b >,$$

$$[< a, b >, A \otimes c] = A \otimes [L_a, L_b]c,$$

$$[< a, b >, < c, d >] = < [L_a, L_b]c, d > + < c, [L_a, L_b]d >,$$

for $a, b, c, d \in \mathcal{T}$, and $A, B \in sl_2(\mathbf{C})$.

Note that

$$[< a, b >, < c, d >] = < [L_a, L_b]c, d > + < c, [L_a, L_b]d >$$

$$= < a(bc) - b(ac), d > + < c, a(bd) - b(ad) >$$

$$= - < (bc)d, a > - < da, bc > + < (ac)d, b > + < db, ac > + < (bd)c, a >$$

$$+ < ca, bd > - < (ad)c, b > - < cb, ad >$$

$$= - < (ad)c - (ac)d, b > - < (bc)d - (bd)c, a >$$

$$= - < [L_c, L_d]a, b > - < a, [L_c, L_d]b >$$

$$= -[< c, d >, < a, b >].$$

Therefore, the map $[\ ,\ ] : \mathcal{G}(\mathcal{T}) \times \mathcal{G}(\mathcal{T}) \to \mathcal{G}(\mathcal{T})$ is a well-defined anti-commutitive bilinear map.

Lemma 2.3.1 Let $A, B, C \in sl_2(\mathbf{C})$, then

$$[[A, B], C] = (B, C)A - (A, C)B,$$

where $(A, B) = 2\mathrm{tr}(AB)$.

Proof. It is routine to check this identity for the base elements $x_{\pm}$, α of $sl_2(\mathbf{C})$. Then the general result follows from this and linearity.

□

Proposition 2.3.2 $\hat{\mathcal{G}}(\mathcal{T})$ is a covering central extension of the TKK algebra $\mathcal{G}(\mathcal{T})$.

Proof. It is easy to see that $\hat{\mathcal{G}}(\mathcal{T})$ is a perfect algebra. Now we define a linear map $\varphi : \hat{\mathcal{G}}(\mathcal{T}) \to \mathcal{G}(\mathcal{T})$ by

$$A \otimes a \mapsto A \otimes a, \quad < a, b > \mapsto [L_a, L_b],$$

for $A \in sl_2(\mathbf{C})$, $a, b \in \mathcal{T}$. The relations (2.3.1) and Lemma 2.2.1 imply that φ is a well-defined surjective map.

To show $\mathrm{Ker}\varphi \subseteq \mathcal{Z}(\hat{\mathcal{G}}(\mathcal{T}))$, we suppose

$$\lambda = \sum_i A_i \otimes a_i + \sum_i < b_i, c_i > \in \mathrm{Ker}\varphi,$$

for some $A_i \in sl_2(\mathbf{C})$, $a_i, b_i, c_i \in \mathcal{T}$. Then, $\varphi(\lambda) = \sum_i A_i \otimes a_i + \sum_i [L_{b_i}, L_{c_i}] = 0$, which implies that $\sum_i A_i \otimes a_i = 0$ and $\sum_i [L_{b_i}, L_{c_i}] = 0$. Therefore the commutation relation (2.3.2) imply that $\lambda = \sum_i < b_i, c_i > \in \mathcal{Z}(\hat{\mathcal{G}}(\mathcal{T}))$, as required.

On account of the commutation relations (2.2.6) and (2.3.2), it remains to show that $\hat{\mathcal{G}}(\mathcal{T})$ satisfies the Jacobi identity. That is, we need to check the following identities (using the notation (2.2.3))

$$[A_1 \otimes a_1, A_2 \otimes a_2, A_3 \otimes a_3] = 0. \tag{2.3.3}$$

$$[A_1 \otimes a_1, A_2 \otimes a_2, < b_1, b_2 >] = 0. \tag{2.3.4}$$

$$[A_1 \otimes a_1, < b_1, b_2 >, < c_1, c_2 >] = 0. \tag{2.3.5}$$

$$[< a_1, b_1 >, < a_2, b_2 >, < a_3, b_3 >] = 0. \tag{2.3.6}$$

for $A_i \in sl_2(\mathbb{C})$, $a_i, b_i, c_i \in \mathcal{T}$.

To show (2.3.3), we compute

$$\begin{aligned}
&[[A_1 \otimes a_1, A_2 \otimes a_2], A_3 \otimes a_3] \\
&= [[A_1, A_2] \otimes a_1a_2 + (A_1, A_2) < a_1, a_2 >, A_3 \otimes a_3] \\
&= [[A_1, A_2], A_3] \otimes (a_1a_2)a_3 + ([A_1, A_2], A_3) < a_1a_2, a_3 > \\
&\quad + (A_1, A_2)A_3 \otimes [L_{a_1}, L_{a_2}]a_3 \\
&= \{(A_2, A_3)A_1 - (A_1, A_3)A_2\} \otimes (a_1a_2)a_3 + ([A_1, A_2], A_3) < a_1a_2, a_3 > \\
&\quad + (A_1, A_2)A_3 \otimes \{a_1(a_2a_3) - a_2(a_1a_3)\},
\end{aligned}$$

where we have used Lemma 2.3.1.

Therefore, by cycling A_i, a_i, and using cancellation, we obtain

$$\begin{aligned}
&[A_1 \otimes a_1, A_2 \otimes a_2, A_3 \otimes a_3] \\
&= ([A_1, A_2], A_3) < a_1a_2, a_3 > + ([A_2, A_3], A_1) < a_2a_3, a_1 > \\
&\quad + ([A_3, A_1], A_2) < a_3a_1, a_2 >
\end{aligned}$$

$$= ([A_1, A_2], A_3)\{< a_1a_2, a_3 > + < a_2a_3, a_1 > + < a_3a_1, a_2 >\} = 0,$$

where we have used (2.3.1) and the fact that the bilinear form $(\ , \ )$ of $sl_2(\mathbb{C})$ is symmetric and invarient.

To show (2.3.4), we have

$$[[A_1 \otimes a_1, A_2 \otimes a_2], < b_1, b_2 >]$$

$$= [[A_1, A_2] \otimes a_1a_2 + (A_1, A_2) < a_1, a_2 >, < b_1, b_2 >]$$

$$= -[A_1, A_2] \otimes [L_{b_1}, L_{b_2}](a_1a_2) + (A_1, A_2)[< a_1, a_2 >, < b_1, b_2 >],$$

and

$$[[A_2 \otimes a_2, < b_1, b_2 >], A_1 \otimes a_1] = [-A_2 \otimes [L_{b_1}, L_{b_2}]a_2, A_1 \otimes a_1]$$

$$= [A_1, A_2] \otimes a_1([L_{b_1}, L_{b_2}]a_2) + (A_1, A_2) < a_1, [L_{b_1}, L_{b_2}]a_2 >,$$

and

$$[[< b_1, b_2 >, A_1 \otimes a_1], A_2 \otimes a_2]$$

$$= [A_1 \otimes [L_{b_1}, L_{b_2}]a_1, A_2 \otimes a_2]$$

$$= [A_1, A_2] \otimes ([L_{b_1}, L_{b_2}]a_1)a_2 + (A_1, A_2) < [L_{b_1}, L_{b_2}]a_1, a_2 > .$$

Therefore, (2.3.4) follows from the above three identities, the commutation relation (2.3.2) and Lemma 2.2.2.

To show (2.3.5), we let $D = [L_{b_1}, L_{b_2}]$, and we compute

$$[[< b_1, b_2 >, A_1 \otimes a_1], < c_1, c_2 >] + [[A_1 \otimes a_1, < c_1, c_2 >], < b_1, b_2 >]$$

$$= [A_1 \otimes [L_{b_1}, L_{b_2}]a_1, < c_1, c_2 > +[-A_1 \otimes [L_{c_1}, L_{c_2}]a_1, < b_1, b_2 >]$$

$$= -A_1 \otimes [L_{c_1}, L_{c_2}]([L_{b_1}, L_{b_2}]a_1) + A_1 \otimes [L_{b_1}, L_{b_2}]([L_{c_1}, L_{c_2}]a_1)$$

$$= A_1 \otimes [D, [L_{c_1}, L_{c_2}]]a_1$$

$$= A_1 \otimes ([L_{Dc_1}, L_{c_2}] + [L_{c_1}, L_{Dc_2}])a_1$$

$$= [< Dc_1, c_2 > + < c_1, Dc_2 >, A_1 \otimes a_1]$$

$$= -[< c_1, c_2 >, < b_1, b_2 >], A_1 \otimes a_1],$$

which proves (2.3.5).

Finally, we prove (2.3.6). Let $D_i = [L_{a_i}, L_{b_i}]$, $i = 1, 2, 3$. We have

$$[[< a_1, b_1 >, < a_2, b_2 >], < a_3, b_3 >]$$

$$= [< D_1 a_2, b_2 > + < a_2, D_1 b_2 >, < a_3, b_3 >]$$

$$=< ([L_{D_1 a_2}, L_{b_2}] + [L_{a_2}, L_{D_1 b_2}])a_3, b_3 >$$

$$+ < a_3, ([L_{D_1 a_2}, L_{b_2}] + [L_{a_2}, L_{D_1 b_2}])b_3 >$$

$$=< [D_1, D_2]a_3, b_3 > + < a_3, [D_1, D_2]b_3 >,$$

and

$$[[< a_2, b_2 >, < a_3, b_3 >], < a_1, b_1 >]$$

$$= [< D_2 a_3, b_3 > + < a_3, D_2 b_3 >, < a_1, b_1 >]$$

$$= - < D_1(D_2 a_3), b_3 > - < D_2 a_3, D_1 b_3 > - < D_1 a_3, D_2 b_3 >$$

$$- < a_3, D_1(D_2 b_3) > .$$

Thus, the above two identities give us

$$[[< a_1, b_1 >, < a_2, b_2 >], < a_3, b_3 >] + [[< a_2, b_2 >, < a_3, b_3 >], < a_1, b_1 >]$$

$$= - < D_2(D_1 a_3), b_3 > - < D_2 a_3, D_1 b_3 > - < a_3, D_2(D_1 b_3) >$$

$$- < D_1 a_3, D_2 b_3 >$$

$$= -[< a_2, b_2 >, < D_1 a_3, b_3 >] - [< a_2, b_2 >, < a_3, D_1 b_3 >]$$

$$= -[< a_2, b_2 >, [< a_1, b_1 >, < a_3, b_3 >]]$$

$$= -[[<< a_3, b_3 >, < a_1, b_1 >], < a_2, b_2 >],$$

as required. This completes the proof of this Proposition.

$\square$

Lemma 2.3.3[MRY] Let $(\mathcal{L}, \pi)$ be a covering central extension of a perfect Lie algebra $\mathcal{G}$. If $\eta : \mathcal{L} \to \mathcal{L}$ is a Lie algebra endomorphism which induces the identity map on $\mathcal{G}$. Then $\eta = \mathrm{id}_{\mathcal{L}}$.

Proof. For $x, y \in \mathcal{G}$, let $\hat{x}, \hat{y} \in \mathcal{L}$, such that $\pi(\hat{x}) = x$, $\pi(\hat{y}) = y$.

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\pi"] \arrow[d, "\eta"'] & \mathcal{G} \arrow[d, equal] \\ \mathcal{L} \arrow[r, "\pi"] & \mathcal{G} \end{tikzcd}$$

Since η induces the identity map on $\mathcal{G}$, $(\pi \circ \eta)(\hat{x}) = x$, and $(\pi \circ \eta)(\hat{y}) = y$. These imply that $\eta(\hat{x}) - \hat{x}$, $\eta(\hat{y}) - \hat{y} \in \mathrm{Ker}\pi$, and $\eta([\hat{x}, \hat{y}]) = [\eta(\hat{x}), \eta(\hat{y})] = [\hat{x}, \hat{y}]$. Therefore, the result follows from this and the perfectness of $\mathcal{L}$.

□

Proposition 2.3.4 $\hat{\mathcal{G}}(\mathcal{T})$ is a universal central extension of the TKK algebra $\mathcal{G}(\mathcal{T})$.

Proof. Let $(\mathcal{L}, \pi)$ be a universal central extension of the perfect Lie algebra $\mathcal{G}(\mathcal{T})$. As $\hat{\mathcal{G}}(\mathcal{T})$ is a covering central extension of $\mathcal{G}(\mathcal{T})$, there exists a unique homomorphism $\psi : \mathcal{L} \to \hat{\mathcal{G}}(\mathcal{T})$ for which $\varphi\psi = \pi$, that is, the following diagram commutes

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\pi"] \arrow[d, "\psi"'] & \mathcal{G}(\mathcal{T}) \arrow[d, equal] \\ \hat{\mathcal{G}}(\mathcal{T}) \arrow[r, "\varphi"] & \mathcal{G}(\mathcal{T}) \end{tikzcd}$$

where the surjective homomorphism $\varphi : \hat{\mathcal{G}}(\mathcal{T}) \to \mathcal{G}(\mathcal{T})$ is given by

$$\varphi : \quad A \otimes a \mapsto A \otimes a, \quad < a, b > \mapsto [L_a, L_b],$$

for $A \in sl_2(\mathbf{C})$, $a, b \in \mathcal{T}$. We have shown, in the proof of Proposition 2.3.2, that $\mathrm{Ker}\varphi \subseteq \mathcal{Z}(\hat{\mathcal{G}}(\mathcal{T}))$. In fact, this implies $\mathrm{Ker}\varphi = \mathcal{Z}(\hat{\mathcal{G}}(\mathcal{T}))$, since $\mathcal{Z}(\mathcal{G}(\mathcal{T})) = \{0\}$.

In what follows, we are going to define a Lie algebra homomorphism $\psi' : \hat{\mathcal{G}}(\mathcal{T}) \to \mathcal{L}$ for which the endomorphisms $\psi\psi' : \hat{\mathcal{G}}(\mathcal{T}) \to \hat{\mathcal{G}}(\mathcal{T})$, and $\psi'\psi : \mathcal{L} \to \mathcal{L}$ induce the identity map on $\mathcal{G}(\mathcal{T})$. Therefore, Lemma 2.3.3 implies that ψ' defines a isomorphism from $\hat{\mathcal{G}}(\mathcal{T})$ to $\mathcal{L}$ as required.

Let $x_\pm(a)$, $\epsilon(a,b)$ be any preimage elements of $x_\pm \otimes a$, $<a,b>$ respectively of the homomorphism ψ, for $a, b \in \mathcal{T}$. As

$$\psi([[x_+(1), x_-(1)],\ x_+(a)]) = \psi(2x_+(a)),$$

we have

$$[[x_+(1), x_-(1)],\ x_+(a)] = 2x_+(a) + 2c_+^a,$$

for some $c_+^a \in \mathcal{Z}(\mathcal{L})$. We replace $x_+(a)$ by $x_+(a) + c_+^a$, and obtain

$$(2.3.7) \qquad [[x_+(1), x_-(1)],\ x_+(a)] = 2x_+(a),$$

and

$$x_+(a) - c_+^a \in \psi^{-1}(x_+ \otimes a), \tag{2.3.8}$$

for $a \in \mathcal{T}$, and some element $c_+^a \in \mathcal{Z}(\mathcal{L})$.

Similarly, we may define $x_-(a)$ for which

$$(2.3.9) \qquad [[x_+(1), x_-(1)],\ x_-(a)] = -2x_-(a),$$

and

$$(2.3.10) \qquad x_-(a) - c_-^a \in \psi^{-1}(x_- \otimes a),$$

for $a \in \mathcal{T}$, and some element $c_-^a \in \mathcal{Z}(\mathcal{L})$.

Claim 1 $\psi : \mathcal{Z}(\mathcal{L}) \to \mathcal{Z}(\hat{\mathcal{G}}(\mathcal{T}))$.

In fact, since $\mathcal{Z}(\mathcal{G}(\mathcal{T})) = \{0\}$, we have $\varphi(\psi(c)) = \pi(c) = 0$, for any $c \in \mathcal{Z}(\mathcal{L})$. This implies $\psi(c) \in \mathrm{Ker}\varphi \subseteq \mathcal{Z}(\hat{\mathcal{G}}(\mathcal{T}))$, as required. This completes the proof of claim 1.

Now, let

$$(2.3.11) \qquad \alpha(a) = [x_+(1), x_-(a)],$$

for $a \in \mathcal{T}$.

By claim 1, we have $\psi(\alpha(a)) = \alpha \otimes a$, for $a \in \mathcal{T}$. This and claim 1 also give us

$$\psi([x_+(a), x_-(b)]) = \psi(\alpha(ab) + 2\psi(\epsilon(a,b)),$$

for $a, b \in \mathcal{T}$. Hence

$$[x_+(a), x_-(b)] = \alpha(ab) + 2\epsilon(a,b) + 2c_{a,b},$$

for some $c_{a,b} \in \mathcal{Z}(\mathcal{L})$. Therefore, replacing $\epsilon(a,b)$ by $\epsilon(a,b) + c_{a,b}$, we obtain

$$[x_+(a), x_-(b)] = \alpha(ab) + 2\epsilon(a,b), \tag{2.3.12}$$

and

$$\epsilon(a,b) - c_{a,b} \in \psi^{-1}(< a, b >), \tag{2.3.13}$$

for $a, b \in \mathcal{T}$, and some $c_{a,b} \in \mathcal{Z}(\mathcal{L})$.

Claim 2 The elements $x_\pm(a), \alpha(a)$ and $\epsilon(a,b)$ for $a, b \in \mathcal{T}$ satisfy the following commutation relations

(i) $[x_\pm(a),\ x_\pm(b)] = 0$,

(ii) $[x_+(a),\ x_-(b)] = \alpha(ab) + 2\epsilon(a,b)$,

(iii) $[\alpha(a),\ x_\pm(b)] = \pm 2x_\pm(ab)$,

(iv) $[\alpha(a),\ \alpha(b)] = 4\epsilon(a,b)$,

(v) $[\epsilon(a,b),\ x_\pm(c)] = x_\pm([L_a, L_b]c)$, $\quad [\epsilon(a,b),\ \alpha(a)] = \alpha([L_a, L_b]c)$,

(vi) $[\epsilon(a,b),\ \epsilon(c,d)] = \epsilon([L_a, L_b]c, d) + \epsilon(c, [L_a, L_b]d)$,

for $a, b, c, d \in \mathcal{T}$.

Proof of claim 2. For (i), note that (2.3.7)(2.3.9) imply

$$[\alpha(1),\ x_\pm(a)] = \pm 2x_\pm(a). \tag{2.3.14}$$

As $\psi([x_+(a), x_+(b)]) = 0$, so $[x_+(a), x_+(b)] \in \mathcal{Z}(\mathcal{L})$. Thus, by the Jacobi identity

$$0 = [[x_+(a), x_+(b)],\ \alpha(1)]$$

$$= -[[x_+(b), \alpha(1)],\ x_+(a)] - [[\alpha(1), x_+(a)],\ x_+(b)]$$

$$= -[-2x_+(b), x_+(a)] - [2x_+(a), x_+(b)] = -4[x_+(a), x_+(b)],$$

which gives $[x_+(a), x_+(b)] = 0$. Similarly, we can prove $[x_-(a), x_-(b)] = 0$, for any $a, b \in \mathcal{T}$.

(ii) follows from (2.3.12). Now we prove (iii). For this purpose, we need the following identities

(a) $[\alpha(1), \alpha(a)] = 0$,

(b) $[\alpha(a), x_\pm(1)] = \pm 2x_\pm(a)$,

(c) $[x_+(a), x_-(1)] = \alpha(a)$,

(d) $\epsilon(1, a) = \epsilon(a, 1) = 0$,

for $a \in \mathcal{T}$.

Indeed,

$$[\alpha(1), \alpha(a)] = [\alpha(1), [x_+(1), x_-(a)]]$$

$$= -[x_+(1), [x_-(a), \alpha(1)]] - [x_-(a), [\alpha(1), x_+(1)]]$$

$$= -[x_+(1), 2x_-(a)] - [x_-(a), 2x_+(1)] = 0.$$

This proves (a). To show (b), we note that

$$[\alpha(a), x_\pm(1)] \mp 2x_\pm(a) \in \mathcal{Z}(\mathcal{L}).$$

Hence

$$2x_+(a) = [\alpha(1), x_+(a)] = [\alpha(1), \frac{1}{2}[\alpha(a), x_+(1)]]$$

$$= -\frac{1}{2}[\alpha(a), [x_+(1), \alpha(1)]] - \frac{1}{2}[x_+(1), [\alpha(1), \alpha(a)]]$$

$$= -\frac{1}{2}[\alpha(a), -2x_+(1)] = [\alpha(a), x_+(1)],$$

as required. Similarly, we have $[\alpha(a), x_-(1)] = -2x_-(a)$, for $a \in \mathcal{T}$.

To show (c), we use (2.3.11)

$$[x_+(a), x_-(1)] = [\frac{1}{2}[\alpha(a), x_+(1)], x_-(1)]$$

$$= -\frac{1}{2}[[x_+(1), x_-(1)], \alpha(a)] - \frac{1}{2}[[x_-(1), \alpha(a)], x_+(1)]$$

$$= -\frac{1}{2}[\alpha(1), \alpha(a)] - \frac{1}{2}[2x_-(a), x_+(1)] = \alpha(a).$$

as required. Finally, (d) follows from (c), (2.3.11) and (ii).

Now we prove (iii). Since $[\alpha(a), x_+(b)] - 2x_+(ab) \in \mathcal{Z}(\mathcal{L})$, we have

$$[[\alpha(a), x_+(b)], \alpha(1)] = [2x_+(ab), \alpha(1)]$$

$$= -4x_+(ab).$$

On the other hand,

$$[[\alpha(a), x_+(b)], \alpha(1)]$$

$$= -[[x_+(b), \alpha(1)], \alpha(a)] - [[\alpha(1), \alpha(a)], x_+(b)]$$

$$= -[-2x_+(b), \alpha(a)] = -2[\alpha(a), x_+(b)].$$

The above two identities give us,

$$[\alpha(a), x_+(b)] = 2x_+(ab).$$

Similarly, we have $[\alpha(a), x_-(b)] = -2x_-(ab)$.

For (iv), we compute

$$[\alpha(a), \alpha(b)] = [[x_+(1), x_-(a)], \alpha(b)]$$

$$= -[[x_-(a), \alpha(b)], x_+(1)] - [[\alpha(b), x_+(1)], x_-(a)]$$

$$= -[2x_-(ab), x_+(1)] - [2x_+(b), x_-(a)]$$

$$= 2\alpha(ab) + 4\epsilon(1, ab) - 2\alpha(ba) - 4\epsilon(b, a)$$

$$= -4\epsilon(b, a).$$

Also, we have $[\alpha(b), \alpha(a)] = -4\epsilon(a, b)$. Thus

(2.3.15) $$\epsilon(a, b) + \epsilon(b, a) = 0,$$

and

$$[\alpha(a), \alpha(b)] = 4\epsilon(a, b).$$

as required.

For (v), we need the following identities

$$[\epsilon(a,b), x_{\pm}(1)] = 0 = [\epsilon(a,b), \alpha(1)],$$

for $a, b \in \mathcal{T}$.

In fact,

$$[\epsilon(a,b), x_+(1)] = [\frac{1}{2}[x_+(a), x_-(b)] - \frac{1}{2}\alpha(ab), x_+(1)]$$

$$= -\frac{1}{2}[[x_-(b), x_+(1)], x_+(a)] - \frac{1}{2}[[x_+(1), x_+(a)], x_-(b)]$$

$$-\frac{1}{2}[\alpha(ab), x_+(1)]$$

$$= \frac{1}{2}[\alpha(b), x_+(a)] - x_+(ab) = 0.$$

similarly, $[\epsilon(a,b), x_-(1)] = 0$.

And $[\epsilon(a,b), \alpha(1)] = 0$ follows from the fact $\alpha(1) = [x_+(1), x_-(1)]$ and the Jacobi identity.

Now we prove (v). Since $[\epsilon(a,b), x_+(c)] - x_+([L_a, L_b]c) \in \Xi(\mathcal{L})$, we have

$$[[\epsilon(a,b), x_+(c)], \alpha(1)] = [[x_+([L_a, L_b]c), \alpha(1)]$$

$$= -2x_+([L_a, L_b]c).$$

On the other hand,

$$[[\epsilon(a,b), x_+(c)], \alpha(1)]$$

$$= -[[x_+(c), \alpha(1)], \epsilon(a,b)] - [[\alpha(1), \epsilon(a,b)], x_+(c)]$$

$$= 2[x_+(c), \epsilon(a,b)].$$

The above two identities give us

$$[\epsilon(a,b), x_+(c)] = x_+([L_a, L_b]c).$$

Similarly, $[\epsilon(a,b), x_-(c)] = x_-([L_a, L_b]c)$.

Moreover, $[\epsilon(a,b),\alpha(c)] = \alpha([L_a, L_b]c)$ follows from the fact $\alpha(c) = [x_+(1), x_-(c)]$, and the Jacobi identity. This completes the proof of (v).

Finally, for (vi), we have

$$[\epsilon(a,b),\ \epsilon(c,d)] = [\epsilon(a,b), \frac{1}{2}[x_+(c), x_-(d)] - \frac{1}{2}\alpha(cd)]$$

$$= -\frac{1}{2}[x_+(c), [x_-(d), \epsilon(a,b)]] - \frac{1}{2}[x_-(d), [\epsilon(a,b), x_+(c)]]$$

$$-\frac{1}{2}[\epsilon(a,b), \alpha(cd)]$$

$$= -\frac{1}{2}[x_+(c), x_-([L_a, L_b]d)]] - \frac{1}{2}[x_-(d), x_+([L_a, L_b]c)]$$

$$-\frac{1}{2}\alpha([L_a, L_b](cd))]$$

$$= \frac{1}{2}\alpha(c([L_a, L_b]d)) + \epsilon(c, [L_a, L_b]d) + \frac{1}{2}\alpha(([L_a, L_b]c)d))$$

$$+\epsilon([L_a, L_b]c, d) - \frac{1}{2}\alpha([L_a, L_b](cd))$$

$$= \epsilon(c, [L_a, L_b]d) + \epsilon([L_a, L_b]c, d),$$

as required, where we have used Lemma 2.2.2.

Claim 3 $\epsilon(a,b)$ satisfies the following identities

(1). $\epsilon(a,b) + \epsilon(b,a) = 0$.

(2). $\epsilon(ab,c) + \epsilon(bc,a) + \epsilon(ca,b) = 0$.

Proof of claim 3: (1) follows from (2.3.15). For (2) we compute

$$\epsilon(ab,c) = \frac{1}{4}[\alpha(ab), \alpha(c)]$$

$$= \frac{1}{4}[[x_+(a), x_-(b)] - 2\epsilon(a,b),\ \ \alpha(c)]$$

$$= -\frac{1}{4}[[x_-(b), \alpha(c)],\ \ x_+(a)]] - \frac{1}{4}[[\alpha(c), x_+(a)],\ \ x_-(b)]$$

$$-\frac{1}{2}[\epsilon(a,b), \alpha(c)]$$

$$= -\frac{1}{4}[2x_-(bc), x_+(a)] - \frac{1}{4}[2x_+(ac), x_-(b)]$$

$$-\frac{1}{2}\alpha([L_a, L_b]c)$$

$$= \frac{1}{2}\{\alpha(a(bc)) + 2\epsilon(a, bc)\} - \frac{1}{2}\{\alpha((ac)b) + 2\epsilon(ac, b)\}$$
$$-\frac{1}{2}\alpha(a(bc) - b(ac))$$
$$= -\epsilon(bc, a) - \epsilon(ac, b),$$

as required.

Now we define a linear map $\psi' : \hat{\mathcal{G}}(\mathcal{T}) \to \mathcal{L}$ by

$$x_\pm \otimes a \mapsto x_\pm(a),$$

$$\alpha \otimes a \mapsto \alpha(a),$$

$$< a, b > \mapsto \epsilon(a, b),$$

for $a, b \in \mathcal{T}$. Then Claim 1 and Claim 2 imply that ψ' is well defined and gives a Lie algebra homomorphism from $\hat{\mathcal{G}}(\mathcal{T})$ to $\mathcal{L}$.

To complete the proof of Proposition 2.3.4, it remains to check that the endomorphisms $\psi \circ \psi'$ of $\hat{\mathcal{G}}(\mathcal{T})$, and $\psi' \circ \psi$ of $\mathcal{L}$ induce the identity map on $\mathcal{G}(\mathcal{T})$. That is, the following diagrams commute

$$\begin{tikzcd} \hat{\mathcal{G}}(\mathcal{T}) \arrow[r, "\varphi"] \arrow[d, "\psi \circ \psi'"'] & \mathcal{G}(\mathcal{T}) \arrow[d, equal] \\ \hat{\mathcal{G}}(\mathcal{T}) \arrow[r, "\varphi"] & \mathcal{G}(\mathcal{T}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} \arrow[r, "\pi"] \arrow[d, "\psi' \circ \psi"'] & \mathcal{G}(\mathcal{T}) \arrow[d, equal] \\ \mathcal{L} \arrow[r, "\pi"] & \mathcal{G}(\mathcal{T}) \end{tikzcd}$$

In other words, we need to show $\varphi = \varphi \circ \psi \circ \psi'$ and $\pi = \pi \circ \psi' \circ \psi$. In fact, we have

$$\varphi \circ \psi \circ \psi'(x_\pm \otimes a) = \varphi \circ \psi(x_\pm(a)) = \varphi(x_\pm \otimes a - c^a_\pm) = \varphi(x_\pm \otimes a),$$

and

$$\varphi \circ \psi \circ \psi'(\alpha \otimes a) = \varphi \circ \psi(\alpha(a)) = \varphi(\alpha \otimes a),$$

and

$$\varphi \circ \psi \circ \psi'(< a, b >) = \varphi \circ \psi(\epsilon(a, b)) = \varphi(< a, b > - c_{a,b}) = \varphi(< a, b >).$$

This implies $\varphi \circ \psi \circ \psi' = \varphi$, as required. Moreover, since $\pi = \varphi \circ \psi$, we have

$$\pi \circ \psi' \circ \psi = (\varphi \circ \psi) \circ \psi' \circ \psi = (\varphi \circ \psi \circ \psi') \circ \psi = \varphi \circ \psi = \pi.$$

as required. This completes the proof of Proposition 2.3.4.

$\square$

§2.4 Connes Cyclic Homology Group $HC_1(\mathcal{T})$

From now on we shall concentrate on the smallest semilattice (non-lattice) case. That is $S = S_0 \cup S_1 \cup S_2 \in \mathbf{R}^2$, where $S_0 = 2\mathbf{Z}\delta_1 + 2\mathbf{Z}\delta_2$, $S_i = S_0 + \delta_i$, $i = 1, 2$, and $\delta_1 = (1, 0)$, $\delta_2 = (0, 1) \in \mathbf{R}^2$.

For $\sigma_i = a_i\delta_1 + b_i\delta_2$, $i = 1, 2$, we denote by $\sigma_1 \cdot \sigma_2$ the inner product of σ_1 with σ_2 in $\mathbf{R}^2$. That is, $\sigma_1 \cdot \sigma_2 = a_1a_2 + b_1b_2$, for $a_i, b_i \in \mathbf{R}$. We set $S^{\perp} := \mathbf{Z}^2 \setminus S = S_0 + \delta_1 + \delta_2$.

It is clear that the Jordan algebra $\mathcal{T}$ has a $\mathbf{Z}^2$-grading, and

$$\mathcal{T} = \oplus_{\sigma \in \mathbf{Z}^2} \mathcal{T}^{\sigma},$$

where

$$\mathcal{T}^{\sigma} = \begin{cases} \mathbf{C}x^{\sigma}, & \text{if } \sigma \in S, \\ \{0\}, & \text{if } \sigma \notin S. \end{cases}$$

Moreover this grading induces a $\mathbf{Z}^2$-grading on the TKK algebra $\mathcal{G}(\mathcal{T})$,

$$\mathcal{G}(\mathcal{T}) = \oplus_{\sigma \in \mathbf{Z}^2} \mathcal{G}^{\sigma}(\mathcal{T}), \tag{2.4.1}$$

where $\mathcal{G}^{\sigma}(\mathcal{T}) = \mathcal{G}^{\sigma}_{+}(\mathcal{T}) + \mathcal{G}^{\sigma}_{0}(\mathcal{T}) + \mathcal{G}^{\sigma}_{-}(\mathcal{T})$, and (Lemma (2.4.6))

$$\mathcal{G}^{\sigma}_{\pm}(\mathcal{T}) = \begin{cases} \mathbf{C}x_{\pm} \otimes x^{\sigma}, & \text{if } \sigma \in S, \\ \{0\}, & \text{if } \sigma \notin S. \end{cases}$$

$$\mathcal{G}^{\sigma}_{0}(\mathcal{T}) = \begin{cases} \mathbf{C}\alpha \otimes x^{\sigma}, & \text{if } \sigma \in S, \\ \mathbf{C}[L_{x^{\delta_1}}, L_{x^{\sigma-\delta_1}}], & \text{if } \sigma \notin S. \end{cases}$$

We define the derivations d_i, $(i = 1, 2)$ of $\mathcal{G}(\mathcal{T})$ by setting

$$d_i y = (\sigma \cdot \delta_i) y, \tag{2.4.2}$$

for $y \in \mathcal{G}^{\sigma}(\mathcal{T})$, and $\sigma \in \mathbf{Z}^2$.

Let $\varepsilon : \mathcal{T} \to \mathbf{C}$ be a $\mathbf{C}$-linear map defined by

$$\varepsilon(x^\sigma) = \begin{cases} 1, & \text{if } \sigma = 0, \\ 0, & \text{if } \sigma \neq 0. \end{cases}$$

Then $\mathcal{G}(\mathcal{T})$ has an invariant symmetric bilinear form (see [AABGP]) $(\cdot,\cdot)$: $\mathcal{G}(\mathcal{T}) \times \mathcal{G}(\mathcal{T}) \to \mathbf{C}$, such that

$$(A \otimes a, B \otimes b) = (A, B)\varepsilon(ab),$$

$$(A \otimes a, D) = 0,$$

$$(D, [L_a, L_b]) = \varepsilon((Da)b),$$

for $a, b \in \mathcal{T}$, $A, B \in sl_2(\mathbf{C})$ and $D \in \mathrm{Inder}(\mathcal{T})$.

We may define the following extended Lie algebra $\check{\mathcal{L}}(\mathcal{T})$ from the TKK algebra $\mathcal{G}(\mathcal{T})$ by setting

$$\check{\mathcal{L}}(\mathcal{T}) = \mathcal{G}(\mathcal{T}) \oplus \mathcal{C} \oplus \mathcal{D},$$

where $\mathcal{C} = \mathbf{C}c_1 \oplus \mathbf{C}c_2$, $\mathcal{D} = \mathbf{C}d_1 \oplus \mathbf{C}d_2$, equipped with the following Lie products

$$[\check{\mathcal{L}}(\mathcal{T}), \mathcal{C}] = \{0\} = [\mathcal{D}, \mathcal{D}],$$

$$[d_i, y] = d_i y, \qquad i = 1, 2, \quad y \in \mathcal{G}(\mathcal{T}),$$

$$[A \otimes a, \ B \otimes b] = [A, B] \otimes ab + (A, B)[L_a, L_b]$$
$$+ \sum_{i=1,2} (d_i(A \otimes a), B \otimes b))c_i,$$

$$[D, A \otimes a] = A \otimes Da,$$

$$[D, [L_a, L_b]] = [L_{Da}, L_b] + [L_a, L_{Db}] + \sum_{i=1,2} (d_i D, [L_a, L_b])c_i,$$

where $A, B \in sl_2(\mathbf{C})$, $a, b \in \mathcal{T}$ and $D \in \mathrm{Inder}(\mathcal{T})$.

The bilinear form $(\cdot,\cdot)$ of $\mathcal{G}(\mathcal{T})$ can be bilinearly extended to $\check{\mathcal{L}}(\mathcal{T})$ by defining

$$(\mathcal{C}, \mathcal{C}) = \{0\} = (\mathcal{D}, \mathcal{D}),$$

$$(c_i, d_j) = \delta_{ij}, \quad i, j = 1, 2.$$

$$(\mathcal{C}, \mathcal{G}(\mathcal{T})) = \{0\} = (\mathcal{D}, \mathcal{G}(\mathcal{T})).$$

We know (see [AABGP]) that the form $(\cdot,\cdot)$ defines an non-degenerate invarient symmetric bilinear form on $\tilde{\mathcal{L}}(\mathcal{T})$, and $\tilde{\mathcal{L}}(\mathcal{T})$ forms an extended affine Lie algebra of type A_1, with the extended affine root system $R \cong \mathbf{Z}^2 \cup (\alpha + S)$, where α is the simple root of A_1.

Let $\mathcal{L}(\mathcal{T}) := [\tilde{\mathcal{L}}(\mathcal{T}), \tilde{\mathcal{L}}(\mathcal{T})]$ be the derived algebra of $\tilde{\mathcal{L}}(\mathcal{T})$. Then

(2.4.3)
$$\mathcal{L}(\mathcal{T}) = \mathcal{G}(\mathcal{T}) \dotplus \mathcal{C}.$$

In fact, $\mathcal{L}(\mathcal{T})$ is a perfect central extension of the TKK algebra $\mathcal{G}(\mathcal{T})$.

We say a linear map $D : \mathcal{T} \to \mathcal{T}$ a derivation of $\mathcal{T}$, if $D(ab) = (Da)b + a(Db)$, for all $a, b \in \mathcal{T}$.

Let $\mathrm{Der}(\mathcal{T})$ be the algebra of derivations of $\mathcal{T}$. Let $\mathrm{Der}(\mathcal{T})_\sigma$ be the derivations of $\mathcal{T}$ which are of homogeneous degree σ, that is, $D : \mathcal{T}^\tau \to \mathcal{T}^{\tau+\sigma}$ for $D \in \mathrm{Der}(\mathcal{T})_\sigma$. If we set $x^\sigma = 0$ for $\sigma \notin S$, then $Dx^\tau \subseteq \mathbf{C}x^{\tau+\sigma}$, for $\sigma, \tau \in \mathbf{Z}^2$, $D \in \mathrm{Der}(\mathcal{T})_\sigma$. In fact, we have

(2.4.4)
$$\mathrm{Der}(\mathcal{T}) = \dotplus_{\sigma\in\mathbf{Z}^2}\mathrm{Der}(\mathcal{T})_\sigma.$$

To prove (2.4.4), we consider a more general case. Let $A = \dotplus_{\alpha\in G} A_\alpha$ be a G-graded, finitely generated algebra over $\mathbf{C}$. We say D is a derivation of A if D is a linear map from A to itself, and satisfies

$$D(ab) = (Da)b + a(Db), \qquad \text{for } a, b \in A.$$

Let $\mathrm{Der}(A)$ be the set of all derivations of A. $D \in \mathrm{Der}(A)$ is called a derivation on A of homogeneous degree α if $DA_\beta \subset A_{\alpha+\beta}$, $\alpha, \beta \in G$. Set

$$\mathrm{Der}(A)_\alpha = \{D \in \mathrm{Der}(A) | \quad \deg D = \alpha\}.$$

Lemma 2.4.1 [F] $\mathrm{Der}(A) = \dotplus_{\alpha\in G}\mathrm{Der}(A)_\alpha$.

Proof. Let $S \subset A$ be a finite set such that S generates A. Let $D \in \mathrm{Der}(A)$. There exist two finite sets $G_0, G_1 \subseteq G$ such that

$$S \subseteq \dot{+}_{\alpha\in G_0} A_\alpha, \qquad D(S) \subseteq \dot{+}_{\alpha\in G_1} A_\alpha.$$

For $\alpha \in G$, let $P_\alpha : A \to A_\alpha$ be the projection map. Put $D_\alpha = \sum_{\beta\in G} P_{\alpha+\beta} \circ D \circ P_\beta$. Then D_α is a well defined linear map of A into itself. Moreover, for $a_\mu \in A_\mu$, $a_\nu \in A_\nu$, we have

$$\begin{aligned} D_\alpha(a_\mu a_\nu) &= \sum_{\beta\in G} P_{\alpha+\beta} \circ D \circ P_\beta(a_\mu a_\nu) \\ &= P_{\alpha+\mu+\nu} \circ D(a_\mu a_\nu) \\ &= P_{\alpha+\mu+\nu}((Da_\mu)a_\nu + a_\mu(Da_\nu)) \\ &= (P_{\alpha+\mu}(Da_\mu))a_\nu + a_\mu(P_{\alpha+\nu}(Da_\nu)) \\ &= D_\alpha(a_\mu)a_\nu + a_\mu(D_\alpha a_\nu). \end{aligned}$$

This implies that $D_\alpha \in \mathrm{Der}(A)_\alpha \subseteq \mathrm{Der}(A)$, for $\alpha \in G$.

Now we let $G_2 = \{\alpha - \beta : \ \alpha \in G_1, \beta \in G_0\} \subseteq G$, let $b \in S$, and obtain

$$D(b) = \sum_{\alpha\in G_1} P_\alpha \circ D(b) = \sum_{\alpha\in G_1}\sum_{\beta\in G_0} P_\alpha \circ D \circ P_\beta(b)$$

$$= \sum_{\alpha\in G_1}\sum_{\beta\in G_0} P_{(\alpha-\beta)+\beta} \circ D \circ P_\beta(b) = \sum_{\gamma\in G_2}\sum_{\beta\in G_0} P_{\gamma+\beta} \circ D \circ P_\beta(b)$$

$$= \sum_{\gamma\in G_2}\sum_{\beta\in G_0} P_{\gamma+\beta} \circ D \circ P_\beta(b) = \sum_{\gamma\in G_2} D_\gamma(b).$$

That is, $D(b) = \sum_{\gamma\in G_2} D_\gamma(b)$, for $b \in S$, which implies $D = \sum_{\gamma\in G_2} D_\gamma$ on A. Therefore,

$$\mathrm{Der}(A) = \dot{+}_{\alpha\in G}\mathrm{Der}(A)_\alpha,$$

as required.

□

Lemma 2.4.2 For $\sigma \in S_0$, $\tau \in S_1 \cup S_2$, we have

$$< x^\sigma, x^\tau >= 0.$$

Proof. If $\tau \in S_1$, we have

$$< x^\sigma, x^\tau > = < x^{\sigma-\delta_2} x^{\delta_2}, x^\tau >$$

$$= - < x^{\delta_2} x^\tau, x^{\sigma-\delta_2} > - < x^\tau x^{\sigma-\delta_2}, x^{\delta_2} > = 0,$$

where in the last identity we have used the fact $x^\alpha x^\beta = 0$, if $\alpha, \beta \in S$ and $\alpha + \beta \notin S$.

Similarly, $< x^\sigma, x^\tau > = 0$ for $\tau \in S_2$.

□

Proposition 2.4.3 Let $\sigma, \tau \in S_0$. We have

$$< x^{\sigma+\delta_i}, x^{\tau-\delta_j} > = < x^{\delta_i}, x^{\sigma+\tau-\delta_j} > + \delta_{ij} \sum_{k=1,2} (\sigma \cdot \delta_k) < x^{\delta_k}, x^{\sigma+\tau-\delta_k} >,$$

for $i, j = 0, 1, 2$, where $\delta_0 := 0$.

Proof. Let $\sigma = 2m\delta_1 + 2n\delta_2$, $\tau = 2k\delta_1 + 2l\delta_2$ for $m, n, k, l \in \mathbf{Z}$. We first suppose that $m \in \mathbf{Z}_+$, and obtain

$$< x^\sigma, x^\tau > = < x^{\sigma-\delta_1} x^{\delta_1}, x^\tau >$$

$$= - < x^{\delta_1} x^\tau, x^{\sigma-\delta_1} > - < x^\tau x^{\sigma-\delta_1}, x^{\delta_1} >$$

$$= < x^{\sigma-\delta_1}, x^{\tau+\delta_1} > + < x^{\delta_1}, x^{\tau+\sigma-\delta_1} >$$

$$= < x^{\sigma-2\delta_1} x^{\delta_1}, x^{\tau+\delta_1} > + < x^{\delta_1}, x^{\tau+\sigma-\delta_1} >$$

$$= - < x^{\delta_1} x^{\tau+\delta_1}, x^{\sigma-2\delta_1} > - < x^{\sigma-2\delta_1} x^{\tau+\delta_1}, x^{\delta_1} > + < x^{\delta_1}, x^{\tau+\sigma-\delta_1} >$$

$$= < x^{\sigma-2\delta_1}, x^{\tau+2\delta_1} > + 2 < x^{\delta_1}, x^{\sigma+\tau-\delta_1} > .$$

Thus, inductively we have from this

$$(2.4.5) \qquad < x^\sigma, x^\tau > = < x^{\sigma-2m\delta_1}, x^{\tau+2m\delta_1} > + 2m < x^{\delta_1}, x^{\sigma+\tau-\delta_1} >,$$

for $m \in \mathbf{Z}_+$, and $\sigma, \tau \in S_0$.

If $m \in -\mathbf{Z}_+$, we have

$$< x^\sigma, x^\tau > = - < x^\tau, x^\sigma > = - < x^{\tau+2m\delta_1} x^{-2m\delta_1}, x^\sigma >$$

$$=< x^{-2m\delta_1}x^{\sigma}, x^{\tau+2m\delta_1} > + < x^{\sigma}x^{\tau+2m\delta_1}, x^{-2m\delta_1} >$$

$$= - < x^{-2m\delta_1}, x^{\sigma+\tau+2m\delta_1} > + < x^{\sigma-2m\delta_1}, x^{\tau+2m\delta_1} >$$

$$= -(-2m) < x^{\delta_1}, x^{\sigma+\tau-\delta_1} > + < x^{\sigma-2m\delta_1}, x^{\tau+2m\delta_1} >$$

$$= 2m < x^{\delta_1}, x^{\sigma+\tau-\delta_1} > + < x^{\sigma-2m\delta_1}, x^{\tau+2m\delta_1} >.$$

where in the second to last step we have used (2.4.5). Therefore, (2.4.5) also hold for $m \in \mathbf{Z}$, and $\sigma, \tau \in S_0$.

Likewise, we have

$$< x^{\sigma}, x^{\tau} >= 2n < x^{\delta_2}, x^{\sigma+\tau-\delta_2} > + < x^{\sigma-2n\delta_2}, x^{\tau+2n\delta_2} > .$$

Thus we obtain

$$(2.4.6) \qquad < x^{\sigma}, x^{\tau} >= 2m < x^{\delta_1}, x^{\sigma+\tau-\delta_1} > +2n < x^{\delta_2}, x^{\sigma+\tau-\delta_2} >.$$

for $\sigma = 2m\delta_1 + 2n\delta_2$, $\tau = 2k\delta_1 + 2l\delta_2$, $m, n, k, l \in \mathbf{Z}$.

Finally, we compute, for $i, j = 0, 1, 2$

$$< x^{\sigma+\delta_i}, x^{\tau-\delta_j} >=< x^{\sigma}x^{\delta_i}, x^{\tau-\delta_j} >$$

$$= - < x^{\delta_i}x^{\tau-\delta_j}, x^{\sigma} > - < x^{\tau-\delta_j}x^{\sigma}, x^{\delta_i} >$$

$$=< x^{\delta_i}, x^{\sigma+\tau-\delta_j} > +\delta_{ij} < x^{\sigma}, x^{\tau} >.$$

where we have used Lemma 2.4.2.

By (2.4.6), we then obtain

$$< x^{\sigma+\delta_i}, x^{\tau-\delta_j} >=< x^{\delta_i}, x^{\sigma+\tau-\delta_j} > +\delta_{ij}2m < x^{\delta_1}, x^{\sigma+\tau-\delta_1} >$$

$$+\delta_{ij}2n < x^{\delta_2}, x^{\sigma+\tau-\delta_2} >$$

$$=< x^{\delta_i}, x^{\sigma+\tau-\delta_j} > +\delta_{ij} \sum_{k=1,2} (\sigma \cdot \delta_k) < x^{\delta_k}, x^{\sigma+\tau-\delta_k} >.$$

as required.

□

Corollary 2.4.4 Let $m, n \in \mathbb{Z}$. Then

$$< x^{m\delta_i}, x^{n\delta_i} >= m\delta_{m+n,0} < x^{\delta_i}, x^{-\delta_i} >,$$

for $i = 1, 2$.

proof. By Proposition 2.4.3 and Lemma 2.4.2, we have

$$< x^{m\delta_i}, x^{n\delta_i} >= m < x^{\delta_i}, x^{(m+n-1)\delta_i} >,$$

and

$$< x^{n\delta_i}, x^{m\delta_i} >= n < x^{\delta_i}, x^{(m+n-1)\delta_i} >.$$

These imply that $< x^{\delta_i}, x^{(m+n-1)\delta_i} >= \delta_{m+n,0} < x^{\delta_i}, x^{-\delta_i} >$, and also $< x^{m\delta_i}, x^{n\delta_i} >= m\delta_{m+n,0} < x^{\delta_i}, x^{-\delta_i} >$, as required.

□

Corollary 2.4.5 Let $\sigma \in S_0$. Then

$$< x^{\delta_1}, x^{\sigma+\delta_2} > + < x^{\delta_2}, x^{\sigma+\delta_1} >= 0.$$

□

Lemma 2.4.6 Let $\sigma, \tau \in S$. We set $v_\rho := [L_{x^{\delta_1}}, L_{x^{\rho-\delta_1}}]$, for $\rho \in S^\perp$. Then we have

$$(2.4.7) \qquad [L_{x^\sigma}, L_{x^\tau}] = \begin{cases} 0, & \text{if } \sigma+\tau \in S, \\ v_{\sigma+\tau}, & \text{if } \sigma+\tau \in S^\perp, \text{ and } \sigma \in S_1, \\ -v_{\sigma+\tau}, & \text{if } \sigma+\tau \in S^\perp, \text{ and } \sigma \in S_2, \end{cases}$$

and $\{v_\rho\}_{\rho \in S^\perp}$ form a basis of Inder($\mathcal{T}$).

Proof. It is straightward to check (2.4.7). Now we show that $\{v_\sigma\}_{\sigma\in S^\perp}$ are linearly independent.

Suppose $\sum_i c_i v_{\sigma_i} = 0$ for some $c_i \in \mathbb{C}$, and distinct $\sigma_i \in S^\perp$, we obtain

$$\sum_i c_i x^{\sigma_i+\delta_1} = -(\sum_i c_i v_{\sigma_i}) x^{\delta_1} = 0.$$

This implies that $c_i = 0$, for all i.

□

Lemma 2.4.7 Let $\sigma, \tau \in S$. Then $< x^\sigma, x^\tau > \in \mathcal{Z}(\hat{\mathcal{G}}(\mathcal{T}))$ if and only if $\sigma + \tau \in S$.

Proof. The Lemma follows from the fact that $[L_{x^\sigma}, L_{x^\tau}] = 0$ if and only if $\sigma+\tau \in S$.

□

Definition 2.4.8 Let J be any Jordan algebra obtained from a semilattice S. Then the Connes cyclic homology group $HC_1(J)$ is defined by

$$HC_1(J) = \{\sum_{i \in I} < a_i, b_i > \mid \sum_{i \in I}[L_{a_i}, L_{b_i}] = 0, \quad a_i, b_i \in J\},$$

where I is any finite index set.

Note: On account of Lemma 2.2.1, we can see that $HC_1(J)$ is well-defined.

Corollary 2.4.9 $HC_1(\mathcal{T}) = \text{span}_{\mathbf{C}}\{< x^{\delta_i}, x^{\sigma-\delta_i} > \mid \sigma \in S_0, \ i = 1, 2\}$.

Proof. This follows from Lemma 2.4.2, Proposition 2.4.3, Lemma 2.4.7 and Lemma 2.4.6.

□

In the rest of this section we study the structure of $HC_1(\mathcal{T})$. First we give some properties of Der$(\mathcal{T})$, which will be used to determine the structure of $HC_1(\mathcal{T})$.

Lemma 2.4.10 Let $\sigma \in \mathbf{Z}^2$. Then

$$\text{Der}(\mathcal{T})_\sigma = \begin{cases} \mathbf{C}x^\sigma d_1 + \mathbf{C}x^\sigma d_2, & \text{if } \sigma \in S_0, \\ \{0\}, & \text{if } \sigma \in S_1 \cup S_2, \\ \mathcal{C}[L_{x^{\delta_1}}, L_{x^{\sigma-\delta_1}}], & \text{if } \sigma \in S^\perp. \end{cases}$$

Proof. Let $\sigma \in \mathbf{Z}^2$. For $D \in \text{Der}(\mathcal{T})_\sigma$, Let

$$Dx^{\delta_i} = t_i x^{\sigma+\delta_i},$$

for some $t_i \in \mathbf{C}$. It is clear that D is determined by t_1, t_2. Thus $\dim \text{Der}(\mathcal{T})_\sigma \leq 2$.

Now for $\sigma \in S^\perp$, and $D \in \text{Der}(\mathcal{T})_\sigma$, we have

$$0 = D(x^{\delta_1} x^{\delta_2}) = t_1 x^{\sigma+\delta_1} x^{\delta_2} + t_2 x^{\delta_1} x^{\sigma+\delta_2}$$

$$= (t_1 + t_2) x^{\sigma+\delta_1+\delta_2}.$$

This implies that $t_1 + t_2 = 0$. and $\dim \mathrm{Der}(\mathcal{T})_\sigma \leq 1$. for $\sigma \in S^\perp$. which then implies that

$$\mathrm{Der}(\mathcal{T})_\sigma = \mathbf{C}[L_{x^{\delta_1}}, L_{x^{\sigma-\delta_1}}].$$

for $\sigma \in S^\perp$.

Now if $\sigma \in S_1$. Then

$$0 = D(x^{\delta_1}x^{\delta_2}) = t_1 x^{\sigma+\delta_1}x^{\delta_2} = t_1 x^{\sigma+\delta_1+\delta_2}.$$

Thus $t_1 = 0$. which gives $Dx^{\delta_i} = 0$. for $i = 1, 2$. Therefore. $D = 0$. that is $\mathrm{Der}(\mathcal{T})_\sigma = \{0\}$ for $\sigma \in S_1$.

Similarly. we have $\mathrm{Der}(\mathcal{T})_\sigma = \{0\}$ for $\sigma \in S_2$. Finally. we suppose that $\sigma \in S_0$. It is clear that $D = x^\sigma(t_1 d_1 + t_2 d_2)$. for some $t_1, t_2 \in \mathbf{C}$. This completes the proof of this Lemma.

□

Corollary 2.4.11 Let $\sigma \in \mathbf{Z}^2$. Then

$$\dim \mathrm{Der}(\mathcal{T})_\sigma = \begin{cases} 2. & \text{if } \sigma \in S_0. \\ 0. & \text{if } \sigma \in S_1 \cup S_2. \\ 1. & \text{if } \sigma \in S^\perp. \end{cases}$$

□

Corollary 2.4.12

$$\mathrm{Der}(\mathcal{T}) = \mathrm{Inder}(\mathcal{T}) \dotplus \mathrm{Outder}(\mathcal{T}).$$

where $\mathrm{Inder}(\mathcal{T}) = [L_\mathcal{T}, L_\mathcal{T}]$. $\mathrm{Outder}(\mathcal{T}) = \mathrm{span}\{x^\sigma d_i \mid \sigma \in S_0,\ i = 1, 2\}$.

□

Let $D \in \mathrm{Der}(\mathcal{T})$. Then D acts on $\mathcal{T} \otimes \mathcal{T}$. via $D(a \otimes b) = (Da) \otimes b + a \otimes (Db)$ for $a, b \in \mathcal{T}$. It is clear that D maps I into itself. where I is defined in section 2.3. Thus D induces an endomorphism on the quotient space $\mathcal{T} \otimes \mathcal{T}/I$. Since

$$HC_1(\mathcal{T}) = \mathrm{span}\{< x^{\delta_i}, x^{\sigma-\delta_i} > \mid i = 1, 2,\ \sigma \in S_0\}.$$

we see that, if $D \in \mathrm{Outder}(\mathcal{T})$, D maps $HC_1(\mathcal{T})(\subseteq \mathcal{T} \otimes \mathcal{T}/I)$ into itself [cf. Proposition 2.4.3]. Moreover, if $D = [L_{x^{\delta_1}}, L_{x^{\sigma-\delta_1}}] \in \mathrm{Inder}(\mathcal{T})$, for some $\sigma \in S^{\perp}$, then

$$D < x^{\delta_1}, x^{\tau-\delta_1} >=< Dx^{\delta_1}, x^{\tau-\delta_1} > + < x^{\delta_1}, Dx^{\tau-\delta_1} >$$

$$=< -x^{\sigma-\delta_1}x^{2\delta_1}, x^{\tau-\delta_1} > + < x^{\delta_1}, -x^{\sigma-\delta_1}x^{\tau} >$$

$$= - < x^{\sigma+\delta_1}, x^{\tau-\delta_1} > - < x^{\delta_1}, x^{\sigma+\tau-\delta_1} >= 0,$$

where we have used Proposition 2.4.3. Similarly, we have $D < x^{\delta_2}, x^{\tau-\delta_2} >= 0$. Thus $\mathrm{Inder}(\mathcal{T})$ acts trivially on $HC_1(\mathcal{T})$.

The subspace $I \subseteq \mathcal{T} \otimes \mathcal{T}$ is generated by homogeneous elements (with respect to the $\mathbf{Z}^2$-grading of $\mathcal{T}$), thus $HC_1(\mathcal{T})$ is also graded by $\mathbf{Z}^2$. In fact, we have

$$\deg < x^{\delta_i}, x^{\sigma-\delta_i} >= \sigma,$$

for $\sigma \in S_0$, $i = 1, 2$. Thus, we have

$$HC_1(\mathcal{T}) = \oplus_{\sigma\in\mathbf{Z}^2} HC_1(\mathcal{T})_{\sigma},$$

where

$$(2.4.8) \qquad HC_1(\mathcal{T})_{\sigma} = \begin{cases} \mathbf{C} < x^{\delta_1}, x^{\sigma-\delta_1} > +\mathbf{C} < x^{\delta_2}, x^{\sigma-\delta_2} >, & \text{if } \sigma \in S_0, \\ \{0\}, & \text{if } \sigma \notin S_0. \end{cases}$$

Lemma 2.4.13 The elements $< x^{\delta_i}, x^{-\delta_i} >$, $i = 1, 2$ form a basis of $HC_1(\mathcal{T})_0$.

Proof. Recall that the algebra $(\mathcal{L}(\mathcal{T}), \varphi_0)$ [see (2.3.3)] is a perfect central extension of the TKK algebra $\mathcal{G}(\mathcal{T})$, where $\varphi_0 : \ \mathcal{L}(\mathcal{T}) \to \mathcal{G}(\mathcal{T})$ is defined by

$$A \otimes a \mapsto A \otimes a, \quad [L_a, L_b] \mapsto [L_a, L_b], \quad c_i \mapsto 0, \qquad (i = 1, 2)$$

for $A \in sl_2(\mathbf{C})$, $a, b \in \mathcal{T}$. Also, recall that we have a unique homomorphism ψ from the universal central extension [see Proposition 2.3.2] $(\hat{\mathcal{G}}(\mathcal{T}), \varphi)$ to $(\mathcal{L}(\mathcal{T}), \varphi_0)$ given by

$$A \otimes a \mapsto A \otimes a, \ < x^{\delta_1}, x^{\rho-\delta_1} >\mapsto [L_{x^{\delta_1}}, L_{x^{\rho-\delta_1}}], \ < x^{\delta_i}, x^{\sigma-\delta_i} >\mapsto \epsilon(x^{\sigma})c_i,$$

where $A \in sl_2(\mathbf{C})$. $a \in \mathcal{T}$. $\sigma \in S_0$. $\rho \in S^{\perp}$.

Therefore $\varphi_0 \circ \psi = \varphi$. This implies that $< x^{\delta_i}, x^{-\delta_i} >$ is a preimage of c_i for $i = 1, 2$. The Lemma then follows from this and the fact that c_i, $i = 1, 2$ are linearly independent elements in $\mathcal{L}(\mathcal{T})$.

□

Proposition 2.4.14 Let $\sigma \in \mathbf{Z}^2$. Then

$$\dim HC_1(\mathcal{T})_\sigma = \begin{cases} 2, & \text{if } \sigma = 0, \\ 1, & \text{if } \sigma \in S_0 \setminus \{0\}, \\ 0, & \text{if } \sigma \notin S_0. \end{cases}$$

Proof. We only need to consider the case for $\sigma \in S_0 \setminus \{0\}$. Note that, by Proposition 2.4.3,

$$(\sigma \cdot \delta_1) < x^{\delta_1}, x^{\sigma-\delta_1} > + (\sigma \cdot \delta_2) < x^{\delta_2}, x^{\sigma-\delta_2} > = 0,$$

for $\sigma \in S_0$. We have $\dim HC_1(\mathcal{T})_\sigma \leq 1$. But, for $C = \sum_{i=1,2} a_i < x^{\delta_i}, x^{\sigma-\delta_i} >$, we have

$$(x^{-\sigma} d_1) C = a_1 < x^{-\sigma+\delta_1}, x^{\sigma-\delta_1} > + a_1(\sigma - \delta_1) \cdot \delta_1 < x^{\delta_1}, x^{-\delta_1} >$$

$$+ a_2(\sigma - \delta_2) \cdot \delta_1 < x^{\delta_2}, x^{-\delta_2} >$$

$$= (a_2 \sigma \cdot \delta_1 - a_1 \sigma \cdot \delta_2) < x^{\delta_2}, x^{-\delta_2} > .$$

This implies that $C \neq 0$, if $a_2 \sigma \cdot \delta_1 - a_1 \sigma \cdot \delta_2 \neq 0$. Thus $\dim HC_1(\mathcal{T})_\sigma = 1$, for $\sigma \in S_0 \setminus \{0\}$.

□

Corollary 2.4.15 Let $\mathcal{T} = \mathcal{T}(S)$, $S = S_0 \cup S_1 \cup S_2 \subseteq \mathbf{R}^2$. Then the Connes Cyclic homology group $HC_1(\mathcal{T})$ is isomorphic to an additive group generated by the elements $C_i(2m, 2n)$, $i = 1, 2$, $m, n \in \mathbf{Z}$, with the relation

$$mC_1(2m, 2n) + nC_2(2m, 2n) = 0, \quad \text{for } m, n \in \mathbf{Z}. \tag{2.4.9}$$

The isomorphism is given by $C_i(2m, 2n) \mapsto < x^{\delta_i}, x^{\sigma-\delta_i} >$, where $\sigma = 2m\delta_1 + 2n\delta_2$.

□

We will identity $< x^{\delta_i}, x^{\sigma-\delta_i} >$ with $C_i(\sigma) := C_i(2m, 2n)$, for $\sigma = 2m\delta_1 + 2n\delta_2$ via this isomorphism.

§2.5 Structure of $\hat{\mathcal{G}}(\mathcal{T})$

From last section we know that $\hat{\mathcal{G}}(\mathcal{T})$ is spanned by the elements $x_\pm \otimes x^\sigma$, $\alpha \otimes x^\sigma$, $< x^{\delta_1}, x^{\rho-\delta_1} >$ and $< x^{\delta_i}, x^{\tau-\delta_i} >$, for $i = 1, 2$, $\tau \in S_0$, $\sigma \in S_0 \cup S_1 \cup S_2$, $\rho \in S^\perp$.

For convenience of notation, we put, for $\sigma = m\delta_1 + n\delta_2$,

(2.5.1) $$x_\pm(\sigma) = x_\pm(m, n) := \begin{cases} x_\pm \otimes x^\sigma, & \text{if } \sigma \in S, \\ 0, & \text{if } \sigma \in S^\perp. \end{cases}$$

$$\alpha(\sigma) = \alpha(m, n) := \begin{cases} \alpha \otimes x^\sigma, & \text{if } \sigma \in S, \\ 2 < x^{\delta_1}, x^{\sigma-\delta_1} >, & \text{if } \sigma \in S^\perp. \end{cases}$$

and

$$C_i(\sigma) = C_i(m, n) := \begin{cases} < x^{\delta_i}, x^{\sigma-\delta_i} >, & \text{if } \sigma \in S_0, \\ 0, & \text{if } \sigma \notin S_0, \end{cases}$$

where $i = 1, 2$, $m, n \in \mathbf{Z}$.

Note that $< x^{\delta_1}, x^{\sigma-\delta_1} >= - < x^{\delta_2}, x^{\sigma-\delta_2} >$, if $\sigma \in S^\perp$, [see Corollary 2.4.5)].

Define

$$\Omega(\tau) = \begin{cases} 0, & \text{if } \tau \in S_0, \\ -1, & \text{if } \tau \in S_1, \\ 1, & \text{if } \tau \in S_2, \end{cases}$$

for $\tau \in S$.

Proposition 2.5.1 The universal central extension $\hat{\mathcal{G}}(\mathcal{T})$ of the TKK algebra $\mathcal{G}(\mathcal{T})$ is spanned by the elements $\{x_\pm(\sigma), \ \alpha(\tau), \ C_i(\rho)\}$, for $i = 1, 2$, $\sigma \in S$, $\tau \in \mathbf{Z}^2 = \mathbf{Z}\delta_1 + \mathbf{Z}\delta_2$, and $\rho \in S_0$, and satisfy the following relations

(R1). For $\sigma, \tau \in S$,

$$[x_\pm(\sigma), x_\pm(\tau)] = 0,$$

$$[x_+(\sigma), x_-(\tau)] = \begin{cases} \Omega(\tau)\alpha(\sigma+\tau), & \text{if } \sigma + \tau \notin S, \\ \alpha(\sigma+\tau) + 2\sum_{i=1,2}(\sigma \cdot \delta_i)C_i(\sigma+\tau), & \text{if } \sigma + \tau \in S, \end{cases}$$

(R2). For $\sigma \in \mathbf{Z}^2$, $\tau \in S$,

$$[\alpha(\sigma), x_\pm(\tau)] = \begin{cases} \pm 2x_\pm(\sigma+\tau), & \text{if } \sigma \in S, \\ 2\Omega(\tau)x_\pm(\sigma+\tau), & \text{if } \sigma \notin S. \end{cases}$$

(R3). For $\sigma, \tau \in \mathbf{Z}^2$,

$$[\alpha(\sigma), \alpha(\tau)] = \begin{cases} 2\Omega(\tau)\alpha(\sigma+\tau), & \text{if } \sigma \notin S,\ \tau \in S, \\ -4\sum_{i=1,2}(\sigma\cdot\delta_i)C_i(\sigma+\tau), & \text{if } \sigma, \tau \notin S, \\ 4\sum_{i=1,2}(\sigma\cdot\delta_i)C_i(\sigma+\tau), & \text{if } \sigma, \tau \in S \text{ and } \sigma+\tau \in S, \\ 2\Omega(\tau)\alpha(\sigma+\tau), & \text{if } \sigma, \tau \in S \text{ and } \sigma+\tau \notin S. \end{cases}$$

(R4). $C_i(\sigma)$, $i = 1, 2$ are central, for $\sigma \in S_0$, and satisfy

$$(\sigma\cdot\delta_1)C_1(\sigma) + (\sigma\cdot\delta_2)C_2(\sigma) = 0.$$

Proof. For (R1), we recall the commutation relation (2.2.2)

$$[A \otimes x^\sigma, B \otimes x^\tau] = [A, B] \otimes x^\sigma x^\tau + (A, B) < x^\sigma, x^\tau >,$$

for $A, B \in sl_2(\mathbf{C})$, $\sigma, \tau \in S$. This implies that

$$[x_\pm(\sigma), x_\pm(\tau)] = 0,$$

for $\sigma, \tau \in S$, and

$$[x_+(\sigma), x_-(\tau)] = \alpha \otimes x^\sigma x^\tau + 2 < x^\sigma, x^\tau > .$$

Let $\sigma = \sigma_0 + \delta_i$, $\tau = \tau_0 - \delta_j$ for some $\sigma_0, \tau_0 \in S_0$, and $i, j = 0, 1, 2$, (recall that $\delta_0 = 0$). By Proposition 2.4.3, we have

$$[x_+(\sigma), x_-(\tau)] = \alpha \otimes x^\sigma x^\tau + 2 < x^{\delta_i}, x^{\sigma_0+\tau_0-\delta_j} >$$

$$+2\delta_{ij}\sum_{k=1,2}(\sigma_0\cdot\delta_k) < x^{\delta_k}, x^{\sigma_0+\tau_0-\delta_k} > .$$

Therefore, if $\sigma + \tau \notin S$, we have

$$[x_+(\sigma), x_-(\tau)] = 2 < x^{\delta_i}, x^{\sigma_0+\tau_0-\delta_j} >= \Omega(\tau)\alpha(\sigma+\tau).$$

If $\sigma + \tau \in S$, we have

$$[x_+(\sigma), x_-(\tau)] = \alpha \otimes x^{\sigma+\tau} + 2 < x^{\delta_1}, x^{\sigma_0+\tau_0-\delta_1} >$$

$$+2 \sum_{k=1,2} (\sigma_0 \cdot \delta_k) < x^{\delta_k}, x^{\sigma_0+\tau_0-\delta_k} > .$$

$$= \alpha(\sigma+\tau) + 2 \sum_{k=1,2} (\sigma \cdot \delta_k) C_k(\sigma+\tau),$$

as required.

For (R2), we divide the proof into two cases.

Case 1. If $\sigma \in S$,

$$[\alpha(\sigma), x_\pm(\tau)] = [\alpha \otimes x^\sigma, x_\pm \otimes x^\tau] = [\alpha, x_\pm] \otimes x^\sigma x^\tau$$

$$= \pm 2 x_\pm \otimes x^\sigma x^\tau = \pm 2 x_\pm(\sigma+\tau).$$

Case 2. If $\sigma \notin S$,

$$[\alpha(\sigma), x_\pm(\tau)] = [2 < x^{\delta_1}, x^{\sigma-\delta_1} >, x_\pm \otimes x^\tau] = 2x_\pm \otimes [L_{x^{\delta_1}}, L_{x^{\sigma-\delta_1}}] x^\tau$$

$$= 2x_\pm \otimes (x^{\delta_1}(x^{\sigma-\delta_1} x^\tau) - x^{\sigma-\delta_1}(x^{\delta_1} x^\tau)) = 2\Omega(\tau) x_\pm(\sigma+\tau).$$

This completes the proof of (R2).

For (R3), we divide the proof into three cases.

Cases 1. If $\sigma \notin S$, $\tau \in S$, we have

$$[\alpha(\sigma), \alpha(\tau)] = [2 < x^{\delta_1}, x^{\sigma-\delta_1} >, \alpha \otimes x^\tau]$$

$$= 2\alpha \otimes [L_{x^{\delta_1}}, L_{x^{\sigma-\delta_1}}] x^\tau = 2\Omega(\tau)\alpha(\sigma+\tau).$$

Case 2. If $\sigma, \tau \notin S$, then

$$[\alpha(\sigma), \alpha(\tau)] = 4[< x^{\delta_1}, x^{\sigma-\delta_1} >, < x^{\delta_1}, x^{\tau-\delta_1} >]$$

$$= 4 < [L_{x^{\delta_1}}, L_{x^{\sigma-\delta_1}}] x^{\delta_1}, x^{\tau-\delta_1} > +4 < x^{\delta_1}, [L_{x^{\delta_1}}, L_{x^{\sigma-\delta_1}}] x^{\tau-\delta_1} >$$

$$= 4 < -x^{\sigma+\delta_1}, x^{\tau-\delta_1} > +4 < x^{\delta_1}, x^{\sigma+\tau-\delta_1} >$$

$$= -4 < x^{(\sigma+\delta_1-\delta_2)+\delta_2}, x^{(\tau-\delta_1+\delta_2)-\delta_2} > +4C_1(\sigma+\tau)$$

$$= -4 < x^{\delta_2}, x^{\sigma+\tau-\delta_2} > -4 \sum_{k=1,2} ((\sigma+\delta_1-\delta_2)\cdot\delta_k) < x^{\delta_k}, x^{\sigma+\tau-\delta_k} >$$

$$+4C_1(\sigma+\tau)$$

$$= -4C_2(\sigma+\tau) - 4 \sum_{k=1,2} ((\sigma+\delta_1-\delta_2)\cdot\delta_k)C_k(\sigma+\tau) + 4C_1(\sigma+\tau)$$

$$= -4 \sum_{k=1,2} (\sigma\cdot\delta_k)C_k(\sigma+\tau).$$

Case 3. If $\sigma, \tau \in S$, so we may assume that

$$\sigma = \sigma_0 + \delta_i, \quad \tau = \tau_0 - \delta_j,$$

for some $\sigma_0, \tau_0 \in S_0$, and $i, j = 0, 1, 2$. We have

$$[\alpha(\sigma), \alpha(\tau)] = [\alpha \otimes x^\sigma, \alpha \otimes x^\tau] = 4 < x^\sigma, x^\tau >$$

$$= 4 < x^{\delta_i}, x^{\sigma_0+\tau_0-\delta_j} > +4\delta_{ij} \sum_{k=1,2} (\sigma_0\cdot\delta_k)C_k(\sigma_0+\tau_0)$$

$$= \begin{cases} 2\Omega(\tau)\alpha(\sigma+\tau), & \text{if } \sigma+\tau \notin S, \\ 4\sum_{k=1,2}(\sigma\cdot\delta_k)C_k(\sigma+\tau), & \text{if } \sigma+\tau \in S. \end{cases}$$

Finally, (R4) follows from Lemma 2.4.7 and Corollary 2.4.15. This completes the proof of this Proposition.

□

Corollary 2.5.2 We may rewrite the relations (R1)-(R4) explicitly as follows: From (R1), we get

$$[x_\pm(m, n), x_\pm(k, l)] = 0.$$

$$[x_+(2m, 2n), x_-(2k, 2l)] = \alpha(2m+2k, 2n+2l) + 4mC_1(2m+2k, 2n+2l)$$

$$+4nC_2(2m+2k, 2n+2l),$$

$$[x_+(2m, 2n), x_-(2k-1, 2l)] = \alpha(2m+2k-1, 2n+2l),$$

$$[x_+(2m, 2n), x_-(2k, 2l-1)] = \alpha(2m+2k, 2n+2l-1),$$

$$[x_+(2m+1, 2n), x_-(2k, 2l)] = \alpha(2m+2k+1, 2n+2l)$$

$$[x_+(2m+1,2n),x_-(2k-1,2l)] = \alpha(2m+2k,2n+2l)+2(2m+1)C_1(2m+2k,2n+2l)$$

$$+4nC_2(2m+2k,2n+2l).$$

$$[x_+(2m+1,2n),x_-(2k,2l-1)] = \alpha(2m+2k+1,2n+2l-1)$$

$$[x_+(2m,2n+1),x_-(2k,2l)] = \alpha(2m+2k,2n+2l+1)$$

$$[x_+(2m,2n+1),x_-(2k-1,2l)] = -\alpha(2m+2k-1,2n+2l+1)$$

$$[x_+(2m,2n+1),x_-(2k,2l-1)] = \alpha(2m+2k,2n+2l)+4mC_1(2m+2k,2n+2l)$$

$$+2(2n+1)C_2(2m+2k,2n+2l).$$

From (R2), we get

$$[\alpha(2m,2n),x_\pm(2k,2l)] = \pm 2x_\pm(2m+2k,2n+2l).$$

$$[\alpha(2m,2n),x_\pm(2k-1,2l)] = \pm 2x_\pm(2m+2k-1,2n+2l).$$

$$[\alpha(2m,2n),x_\pm(2k,2l-1)] = \pm 2x_\pm(2m+2k,2n+2l-1).$$

$$[\alpha(2m,2n+1),x_\pm(2k,2l)] = \pm 2x_\pm(2m+2k,2n+2l+1).$$

$$[\alpha(2m,2n+1),x_\pm(2k-1,2l)] = 0.$$

$$[\alpha(2m,2n+1),x_\pm(2k,2l-1)] = \pm 2x_\pm(2m+2k,2n+2l).$$

$$[\alpha(2m+1,2n),x_\pm(2k,2l)] = \pm 2x_\pm(2m+2k+1,2n+2l).$$

$$[\alpha(2m+1,2n),x_\pm(2k-1,2l)] = \pm 2x_\pm(2m+2k,2n+2l).$$

$$[\alpha(2m+1,2n),x_\pm(2k,2l-1)] = 0.$$

$$[\alpha(2m+1,2n+1),x_\pm(2k,2l)] = 0.$$

$$[\alpha(2m+1,2n+1),x_\pm(2k-1,2l)] = -2x_\pm(2m+2k,2n+2l+1).$$

$$[\alpha(2m+1,2n+1),x_\pm(2k,2l-1)] = 2x_\pm(2m+2k+1,2n+2l).$$

From (R3), we get

$$[\alpha(2m,2n),\alpha(2k,2l)] = 8mC_1(2m+2k,2n+2l)+8nC_2(2m+2k,2n+2l).$$

$$[\alpha(2m,2n),\alpha(2k,2l-1)]=0,$$

$$[\alpha(2m,2n),\alpha(2k-1,2l)]=0,$$

$$[\alpha(2m,2n),\alpha(2k-1,2l-1)]=0,$$

$$[\alpha(2m+1,2n),\alpha(2k,2l-1)]=2\alpha(2m+2k+1,2n+2l-1),$$

$$[\alpha(2m+1,2n),\alpha(2k-1,2l)]=4(2m+1)C_1(2m+2k,2n+2l)+8nC_2(2m+2k,2n+2l),$$

$$[\alpha(2m+1,2n),\alpha(2k-1,2l-1)]=2\alpha(2m+2k,2n+2l-1),$$

$$[\alpha(2m,2n+1),\alpha(2k,2l-1)]=8mC_1(2m+2k,2n+2l)+4(2n+1)C_2(2m+2k,2n+2l),$$

$$[\alpha(2m,2n+1),\alpha(2k-1,2l-1)]=-2\alpha(2m+2k-1,2n+2l),$$

$$[\alpha(2m+1,2n+1),\alpha(2k-1,2l-1)]=-4(2m+1)C_1(2m+2k,2n+2l)$$

$$-4(2n+1)C_2(2m+2k,2n+2l).$$

Finally, from (R4), we get

$C_i(2m,2n)$, for $i=1,2$, are central, and $mC_1(2m,2n)+nC_2(2m,2n)=0$,

for all $m,n\in\mathbf{Z}$.

□

In fact, we can reformulate the above structure identities in terms of formal power series in commuting formal variables. For this purpose, we define formal power series in z with coefficients in $\hat{\mathcal{G}}(\mathcal{T})$,

$$C_i(z,m)=\sum_{j\in\mathbf{Z}}C_i(j,m)z^{-j},$$

$$x_\pm(z,m)=\sum_{j\in\mathbf{Z}}x_\pm(j,m)z^{-j},$$

$$\alpha(z,m)=\sum_{j\in\mathbf{Z}}\alpha(j,m)z^{-j},$$

for $m\in\mathbf{Z}$, $i=1,2$.

Recall that we have $C_i(j,m)=0$ if $j\delta_i+m\delta_2\notin S_0$, and $x_\pm(j,m)=0$ if $j\delta_1+m\delta_2\notin S$.

Proposition 2.5.3 The commutation relations, given in Proposition 2.4.3, of $\hat{\mathcal{G}}(\mathcal{T})$ are completely determined by the following identities of formal power series

$$[x_{\pm}(z_1, m), x_{\pm}(z_2, n)] = 0. \tag{2.5.2}$$

$$[x_+(z_1, 2m), x_-(z_2, 2n)] = \alpha(z_2, 2m+2n)\delta(\frac{z_2}{z_1}) + 2C_1(z_2, 2m+2n)(D\delta)(\frac{z_2}{z_1}) + 4mC_2(z_2, 2m+2n)\delta(\frac{z_2}{z_1}). \tag{2.5.3}$$

$$[x_+(z_1, 2m), x_-(z_2, 2n-1)] = \frac{1}{2}\alpha(z_2, 2m+2n-1)\delta(\frac{z_2}{z_1}) + \frac{1}{2}\alpha(-z_2, 2m+2n-1)\delta(-\frac{z_2}{z_1}). \tag{2.5.4}$$

$$[x_+(z_1, 2m+1), x_-(z_2, 2n)] = \frac{1}{2}\alpha(-z_2, 2m+2n+1)(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1})). \tag{2.5.5}$$

(2.5.6)

$$[x_+(z_1, 2m+1), x_-(z_2, 2n-1)] = \frac{1}{4}(\alpha(z_2, 2m+2n)+\alpha(-z_2, 2m+2n))(\delta(\frac{z_2}{z_1})+\delta(-\frac{z_2}{z_1})) + C_1(z_2, 2m+2n)((D\delta)(\frac{z_2}{z_1}) + (D\delta)(-\frac{z_2}{z_1})) + (2m+1)C_2(z_2, 2m+2n)(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1})).$$

$$[\alpha(z_1, 2m), x_{\pm}(z_2, 2n)] = \pm 2x_{\pm}(z_2, 2m+2n)\delta(\frac{z_2}{z_1}). \tag{2.5.7}$$

$$[\alpha(z_1, 2m), x_{\pm}(z_2, 2n-1)] = \pm x_{\pm}(z_2, 2m+2n-1)(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1})). \tag{2.5.8}$$

$$(2.5.9) \qquad [\alpha(z_1, 2m+1), x_\pm(z_2, 2n)] = \pm 2x_\pm(z_2, 2m+2n+1)\delta(\mp\frac{z_2}{z_1}).$$

$$(2.5.10) \qquad [\alpha(z_1, 2m+1), x_\pm(z_2, 2n-1)] = \pm x_\pm(z_2, 2m+2n)\delta(\pm\frac{z_2}{z_1})$$

$$\pm x_\pm(-z_2, 2m+2n)\delta(\mp\frac{z_2}{z_1}).$$

$$(2.5.11) \quad [\alpha(z_1, 2m), \alpha(z_2, 2n)] = 4C_1(z_2, 2m+2n)(D\delta)(\frac{z_2}{z_1}) + 8mC_2(z_2, 2m+2n)\delta(\frac{z_2}{z_1}).$$

$$(2.5.12) \qquad [\alpha(z_1, 2m), \alpha(z_2, 2n-1)] = \alpha(z_2, 2m+2n-1)(\delta(\frac{z_2}{z_1}) - \delta(-\frac{z_2}{z_1})).$$

$$(2.5.13) \quad [\alpha(z_1, 2m+1), \alpha(z_2, 2n-1)] = (\alpha(-z_2, 2m+2n) - \alpha(z_2, 2m+2n))\delta(-\frac{z_2}{z_1})$$

$$+4C_1(z_2, 2m+2n)(D\delta)(-\frac{z_2}{z_1}) + 4(2m+1)C_2(z_2, 2m+2n)\delta(-\frac{z_2}{z_1}).$$

$$(2.5.14) \qquad [C_i(z_1, 2m), \hat{\mathcal{G}}(\mathcal{T})] = 0, \quad \text{and} \quad D_z C_1(z, 2n) = 2nC_2(z, 2n),$$

for $m, n \in \mathbf{Z}$, $i = 1, 2$, where $D_z = z\frac{\partial}{\partial z}$.

Proof. (2.5.2) is clear. For (2.5.3), we have

$$[x_+(z_1, 2m), x_-(z_2, 2n)] = \sum_{i,j\in\mathbf{Z}} z_1^{-i}z_2^{-j}[x_+(i, 2m), x_-(j, 2n)]$$

$$= \sum_{i,j\in 2\mathbf{Z}} z_1^{-i}z_2^{-j}\{\alpha(i+j, 2m+2n) + 2iC_1(i+j, 2m+2n) + 4mC_2(i+j, 2m+2n)\}$$

$$+ \sum_{i,j\in 2\mathbf{Z}+1} z_1^{-i}z_2^{-j}\{\alpha(i+j, 2m+2n) + 2iC_1(i+j, 2m+2n) + 4mC_2(i+j, 2m+2n)\}$$

$$+\sum_{i\in 2\mathbf{Z},j\in 2\mathbf{Z}+1} z_1^{-i}z_2^{-j}\{\alpha(i+j,2m+2n)\}$$

$$+\sum_{i\in 2\mathbf{Z}+1,j\in 2\mathbf{Z}} z_1^{-i}z_2^{-j}\{\alpha(i+j,2m+2n)\}$$

$$=\sum_{j\in 2\mathbf{Z}}\alpha(j,2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}}(\frac{z_2}{z_1})^i+2\sum_{j\in 2\mathbf{Z}}C_1(j,2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}}i(\frac{z_2}{z_1})^i$$

$$+4m\sum_{j\in 2\mathbf{Z}}C_2(j,2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}}(\frac{z_2}{z_1})^i$$

$$+\sum_{j\in 2\mathbf{Z}}\alpha(j,2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}+1}(\frac{z_2}{z_1})^i+2\sum_{j\in 2\mathbf{Z}}C_1(j,2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}+1}i(\frac{z_2}{z_1})^i$$

$$+4m\sum_{j\in 2\mathbf{Z}}C_2(j,2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}+1}(\frac{z_2}{z_1})^i$$

$$+\sum_{j\in 2\mathbf{Z}+1}\alpha(j,2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}}(\frac{z_2}{z_1})^i+\sum_{j\in 2\mathbf{Z}+1}\alpha(j,2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}+1}(\frac{z_2}{z_1})^i$$

$$=\alpha(z_2,2m+2n)\delta(\frac{z_2}{z_1})+2C_1(z_2,2m+2n)(D\delta)(\frac{z_2}{z_1})+4mC_2(z_2,2m_2n)\delta(\frac{z_2}{z_1}).$$

For (2.5.4), we have

$$[x_+(z_1,2m),x_-(z_2,2n-1)]=\sum_{i\in\mathbf{Z},j\in 2\mathbf{Z}} z_1^{-i}z_2^{-j}[x_+(i,2m),x_-(j,2n-1)]$$

$$=\sum_{i,j\in 2\mathbf{Z}} z_1^{-i}z_2^{-j}\{\alpha(i+j,2m+2n-1)\}+\sum_{i\in 2\mathbf{Z}+1,j\in 2\mathbf{Z}} z_1^{-i}z_2^{-j}\{\alpha(i+j,2m+2n-1)\}$$

$$=\sum_{j\in 2\mathbf{Z}}\alpha(j,2m+2n-1)z_2^{-j}\sum_{i\in 2\mathbf{Z}}(\frac{z_2}{z_1})^i+\sum_{j\in 2\mathbf{Z}+1}\alpha(j,2m+2n-1)z_2^{-j}\sum_{i\in 2\mathbf{Z}+1}(\frac{z_2}{z_1})^i$$

$$=\frac{1}{4}(\alpha(z_2,2m+2n-1)+\alpha(-z_2,2m+2n-1))(\delta(\frac{z_2}{z_1})+\delta(-\frac{z_2}{z_1}))$$

$$+\frac{1}{4}(\alpha(z_2,2m+2n-1)-\alpha(-z_2,2m+2n-1))(\delta(\frac{z_2}{z_1})-\delta(-\frac{z_2}{z_1}))$$

$$=\frac{1}{2}\alpha(z_2,2m+2n-1)\delta(\frac{z_2}{z_1})+\frac{1}{2}\alpha(-z_2,2m+2n-1)\delta(-\frac{z_2}{z_1}).$$

For (2.5.5), we have

$$[x_+(z_1,2m+1),x_-(z_2,2n)]=\sum_{i\in 2\mathbf{Z},j\in\mathbf{Z}} z_1^{-i}z_2^{-j}[x_+(i,2m+1),x_-(j,2n)]$$

$$=\sum_{i,j\in 2\mathbf{Z}} z_1^{-i}z_2^{-j}\{\alpha(i+j,2m+2n+1)\}+\sum_{i\in 2\mathbf{Z},j\in 2\mathbf{Z}+1} z_1^{-i}z_2^{-j}\{-\alpha(i+j,2m+2n+1)\}$$

$$= \sum_{j\in 2\mathbf{Z}} \alpha(j, 2m+2n+1) z_2^{-j} \sum_{i\in 2\mathbf{Z}} (\frac{z_2}{z_1})^i - \sum_{j\in 2\mathbf{Z}+1} \alpha(j, 2m+2n+1) z_2^{-j} \sum_{i\in 2\mathbf{Z}} (\frac{z_2}{z_1})^i$$

$$= \frac{1}{2}\alpha(-z_2, 2m+2n+1)(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1})).$$

For (2.5.6), we have

$$[x_+(z_1, 2m+1), x_-(z_2, 2n-1)] = \sum_{i,j\in 2\mathbf{Z}} z_1^{-i} z_2^{-j} [x_+(i, 2m+1), x_-(j, 2n-1)]$$

$$= \sum_{i,j\in 2\mathbf{Z}} z_1^{-i} z_2^{-j} \{\alpha(i+j, 2m+2n) + 2iC_1(i+j, 2m+2n) + 2(2m+1)C_2(i+j, 2m+2n)\}$$

$$= \sum_{j\in 2\mathbf{Z}} \alpha(j, 2m+2n) z_2^{-j} \sum_{i\in 2\mathbf{Z}} (\frac{z_2}{z_1})^i + 2\sum_{j\in 2\mathbf{Z}} C_1(j, 2m+2n) z_2^{-j} \sum_{i\in 2\mathbf{Z}} i(\frac{z_2}{z_1})^i$$

$$+2(2m+1)\sum_{j\in 2\mathbf{Z}} C_2(j, 2m+2n) z_2^{-j} \sum_{i\in 2\mathbf{Z}} (\frac{z_2}{z_1})^i$$

$$= \frac{1}{4}(\alpha(z_2, 2m+2n) + \alpha(-z_2, 2m+2n))(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1}))$$

$$+C_1(z_2, 2m+2n)((D\delta)(\frac{z_2}{z_1}) + (D\delta)(-\frac{z_2}{z_1}))$$

$$+(2m+1)C_2(z_2, 2m+2n)(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1})).$$

For (2.5.7), we have

$$[\alpha(z_1, 2m), x_\pm(z_2, 2n)] = \sum_{i,j\in \mathbf{Z}} z_1^{-i} z_2^{-j} [\alpha(i, 2m), x_\pm(j, 2n)]$$

$$= \sum_{i,j\in \mathbf{Z}} z_1^{-i} z_2^{-j} \{\pm 2x_\pm(i+j, 2m+2n)\}$$

$$= \pm 2\sum_{j\in \mathbf{Z}} x_\pm(j, 2m+2n) z_2^{-j} \sum_{i\in \mathbf{Z}} (\frac{z_2}{z_1})^i$$

$$= \pm 2x_\pm(z_2, 2m+2n)\delta(\frac{z_2}{z_1}).$$

For (2.5.8), we have

$$[\alpha(z_1, 2m), x_\pm(z_2, 2n-1)] = \sum_{i\in \mathbf{Z}, j\in 2\mathbf{Z}} z_1^{-i} z_2^{-j} [\alpha(i, 2m), x_\pm(j, 2n-1)]$$

$$= \sum_{i,j\in 2\mathbf{Z}} z_1^{-i} z_2^{-j} \{\pm 2x_\pm(i+j, 2m+2n-1)\}$$

$$= \pm 2 \sum_{j \in 2\mathbf{Z}} x_\pm(j, 2m+2n-1) z_2^{-j} \sum_{i \in 2\mathbf{Z}} (\frac{z_2}{z_1})^i$$

$$= \pm x_\pm(z_2, 2m+2n-1)(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1}).$$

For (2.5.9), we have

$$[\alpha(z_1, 2m+1), x_\pm(z_2, 2n)] = \sum_{i,j \in \mathbf{Z}} z_1^{-i} z_2^{-j} [\alpha(i, 2m+1), x_\pm(j, 2n)]$$

$$= \sum_{i,j \in 2\mathbf{Z}} z_1^{-i} z_2^{-j} \{\pm 2x_\pm(i+j, 2m+2n+1)\} + \sum_{i,j \in 2\mathbf{Z}+1} z_1^{-i} z_2^{-j} \{-2x_\pm(i+j, 2m+2n+1)\}$$

$$= \pm 2 \sum_{j \in 2\mathbf{Z}} x_\pm(j, 2m+2n+1) z_2^{-j} \sum_{i \in 2\mathbf{Z}} (\frac{z_2}{z_1})^i - 2 \sum_{j \in 2\mathbf{Z}} x_\pm(j, 2m+2n+1) z_2^{-j} \sum_{i \in 2\mathbf{Z}+1} (\frac{z_2}{z_1})^i$$

$$= \pm 2x_\pm(z_2, 2m+2n+1)\delta(\mp\frac{z_2}{z_1}).$$

For (2.5.10), we have

$$[\alpha(z_1, 2m+1), x_\pm(z_2, 2n-1)] = \sum_{i \in \mathbf{Z}, j \in 2\mathbf{Z}} z_1^{-i} z_2^{-j} [\alpha(i, 2m+1), x_\pm(j, 2n-1)]$$

$$= \sum_{i,j \in 2\mathbf{Z}} z_1^{-i} z_2^{-j} \{\pm 2x_\pm(i+j, 2m+2n)\} + \sum_{i \in 2\mathbf{Z}+1, j \in 2\mathbf{Z}} z_1^{-i} z_2^{-j} \{2x_\pm(i+j, 2m+2n)\}$$

$$= \pm 2 \sum_{j \in 2\mathbf{Z}} x_\pm(j, 2m+2n) z_2^{-j} \sum_{i \in 2\mathbf{Z}} (\frac{z_2}{z_1})^i + 2 \sum_{j \in 2\mathbf{Z}+1} x_\pm(j, 2m+2n) z_2^{-j} \sum_{i \in 2\mathbf{Z}+1} (\frac{z_2}{z_1})^i$$

$$= \pm \frac{1}{2}(x_\pm(z_2, 2m+2n) + x_\pm(-z_2, 2m+2n))(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1}))$$

$$+ \frac{1}{2}(x_\pm(z_2, 2m+2n) - x_\pm(-z_2, 2m+2n))(\delta(\frac{z_2}{z_1}) - \delta(-\frac{z_2}{z_1}))$$

$$= \pm x_\pm(z_2, 2m+2n)\delta(\pm\frac{z_2}{z_1}) \pm x_\pm(-z_2, 2m+2n)\delta(\mp\frac{z_2}{z_1}).$$

For (2.5.11), we have

$$[\alpha(z_1, 2m), \alpha(z_2, 2n)] = \sum_{i,j \in \mathbf{Z}} z_1^{-i} z_2^{-j} [\alpha(i, 2m), \alpha(j, 2n)]$$

$$= \sum_{i,j \in 2\mathbf{Z}} z_1^{-i} z_2^{-j} \{4iC_1(i+j, 2m+2n) + 8mC_2(i+j, 2m+2n)\}$$

$$+ \sum_{i,j \in 2\mathbf{Z}+1} z_1^{-i} z_2^{-j} \{4iC_1(i+j, 2m+2n) + 8mC_2(i+j, 2m+2n)\}$$

$$= 4C_1(z_2, 2m+2n)\sum_{i\in 2\mathbf{Z}} i(\frac{z_2}{z_1})^i + 8mC_2(z_2, 2m+2n)\sum_{i\in 2\mathbf{Z}}(\frac{z_2}{z_1})^i$$

$$+4C_1(z_2, 2m+2n)\sum_{i\in 2\mathbf{Z}+1} i(\frac{z_2}{z_1})^i + 8mC_2(z_2, 2m+2n)\sum_{i\in 2\mathbf{Z}+1}(\frac{z_2}{z_1})^i$$

$$= 4C_1(z_2, 2m+2n)(D\delta)(\frac{z_2}{z_1}) + 8mC_2(z_2, 2m+2n)\delta(\frac{z_2}{z_1}).$$

For (2.5.12), we have

$$[\alpha(z_1, 2m), \alpha(z_2, 2n-1)] = \sum_{i,j\in\mathbf{Z}} z_1^{-i}z_2^{-j}[\alpha(i, 2m), \alpha(j, 2n-1)]$$

$$= \sum_{i\in 2\mathbf{Z}+1, j\in\mathbf{Z}} z_1^{-i}z_2^{-j}\{2\alpha(i+j, 2m+2n-1)\}$$

$$= 2\sum_{j\in\mathbf{Z}} \alpha(j, 2m+2n-1)z_2^{-j}\sum_{i\in 2\mathbf{Z}+1}(\frac{z_2}{z_1})^i$$

$$= \alpha(z_2, 2m+2n-1)(\delta(\frac{z_2}{z_1}) - \delta(-\frac{z_2}{z_1})).$$

For (2.5.13), we have

$$[\alpha(z_1, 2m+1), \alpha(z_2, 2n-1)] = \sum_{i,j\in\mathbf{Z}} z_1^{-i}z_2^{-j}[\alpha(i, 2m+1), \alpha(j, 2n-1)]$$

$$= \sum_{i,j\in 2\mathbf{Z}} z_1^{-i}z_2^{-j}\{4iC_1(i+j, 2m+2n) + 4(2m+1)C_2(i+j, 2m+2n)\}$$

$$+ \sum_{i,j\in 2\mathbf{Z}+1} z_1^{-i}z_2^{-j}\{-4iC_1(i+j, 2m+2n) - 4(2m+1)C_2(i+j, 2m+2n)\}$$

$$+ \sum_{i\in 2\mathbf{Z}, j\in 2\mathbf{Z}+1} z_1^{-i}z_2^{-j}\{-2\alpha(i+j, 2m+2n)\}$$

$$+ \sum_{i\in 2\mathbf{Z}+1, j\in 2\mathbf{Z}} z_1^{-i}z_2^{-j}\{2\alpha(i+j, 2m+2n)\}$$

$$= 4C_1(z_2, 2m+2n)\sum_{i\in 2\mathbf{Z}} i(\frac{z_2}{z_1})^i + 4(2m+1)C_2(z_2, 2m+2n)\sum_{i\in 2\mathbf{Z}}(\frac{z_2}{z_1})^i$$

$$-4C_1(z_2, 2m+2n)\sum_{i\in 2\mathbf{Z}+1} i(\frac{z_2}{z_1})^i - 4(2m+1)C_2(z_2, 2m+2n)\sum_{i\in 2\mathbf{Z}+1}(\frac{z_2}{z_1})^i$$

$$-2\sum_{j\in 2\mathbf{Z}+1}\alpha(j, 2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}}(\frac{z_2}{z_1})^i$$

$$+2\sum_{j\in 2\mathbf{Z}+1}\alpha(j, 2m+2n)z_2^{-j}\sum_{i\in 2\mathbf{Z}+1}(\frac{z_2}{z_1})^i$$

$$= 4C_1(z_2, 2m+2n)(D\delta)(-\frac{z_2}{z_1}) + 4(2m+1)C_2(z_2, 2m+2n)\delta(-\frac{z_2}{z_1})$$
$$-(\alpha(z_2, 2m+2n) - \alpha(-z_2, 2m+2n))\delta(-\frac{z_2}{z_1}).$$

Finally, for (2.5.14), we note that

$$D_z C_1(z, 2n) = D_z \sum_{j\in 2\mathbf{Z}} C_1(j, 2n) z^{-j} = -\sum_{j\in 2\mathbf{Z}} jC_1(j, 2n) z^{-j}$$
$$= \sum_{j\in 2\mathbf{Z}} (2n) C_1(j, 2n) z^{-j} = 2nC_2(z, 2n),$$

as required.

Chapter 3

Vertex Operator Representation of the Universal Central Extension of the TKK Algebra

§ 3.1 Introduction

In this chapter we construct a vertex operator representation for the universal central extension $\hat{\mathcal{G}}(\mathcal{T})$ of the TKK algebra $\mathcal{G}(\mathcal{T})$. We start with a nondegenerate even lattice Γ to form a Heisenberg algebra $\mathcal{H}$. We construct the full Fock space $V := \mathbb{C}[\Gamma] \otimes \mathcal{S}(\mathcal{H}^-) \otimes \Lambda(\mathcal{W}^-)$, where $\mathbb{C}[\Gamma]$ is a twisted group algebra with a two-cocycle $\epsilon: \ \Gamma \times \Gamma \to \{\pm 1\}$, and $\Lambda(\mathcal{W}^-)$ is the standard irreducible representation of a Clifford algebra $\mathcal{W}$. We define vertex operators $x_\pm(z, n)$, $\alpha(z, n)$, and $C_1(z, 2n)$, $C_2(z, 2n)$, for $n \in \mathbf{Z}$, such that their component operators act on V. Our main result says that these component operators span a Lie algebra which is isomorphic to $\hat{\mathcal{G}}(\mathcal{T})$.

In the next section we first construct the full Fock space V, and then give some basic identities of formal power series which we will make use of. We close this section by stating the main result (Theorem 3.2.8) of this chapter. Section 3.3 is devoted to the proof of Theorem 3.2.8. In section 3.4 we give a gradation to $\hat{\mathcal{G}}(\mathcal{T})$. We then extend the vertex operator representation of $\hat{\mathcal{G}}(\mathcal{T})$ on V to a representation of the extended TKK algebra $\tilde{\mathcal{G}}(\mathcal{T}) := \hat{\mathcal{G}}(\mathcal{T}) \oplus \mathcal{D}$, where $\mathcal{D} = \mathbb{C}d_1 + \mathbb{C}d_2$ [cf. (3.4.2)]. In the final section, §3.5, we state some corollaries of Theorem 3.2.8, which again give realizations of the toroidal Lie algebra of type A_1 (with two variables) in both the homogenerous and principal pictures.

§3.2 Fock Space and Vertex Operators

Let α, c, d and c_0 be symbols. We form a lattice $\Gamma := \mathbf{Z}\alpha + \mathbf{Z}c + \mathbf{Z}d$ of the real vector space $\mathbf{R} \otimes_{\mathbf{Z}} \Gamma$, with a nondegenerate symmtric bilinear form $(\cdot,\cdot)$ defined by setting

$$(\alpha, \alpha) = 4 = (c, d),$$

and

$$(\alpha, c) = (\alpha, d) = (c, c) = (d, d) = 0.$$

We extend this form bilinearly over $\mathbf{C}$ to $H := \mathbf{C} \otimes_{\mathbf{Z}} \Gamma$. Let $\alpha(m)$, $c(2m)$, $d(2m)$ be a linear copy of α, c, d respectively for $m \in \mathbf{Z}$. Now we form a Heisenberg algebra

$$\mathcal{H} = \mathrm{span}_{\mathbf{C}}\{\alpha(m),\ c(2m),\ d(2m),\ c_0 |\ m \in \mathbf{Z} \setminus \{0\}\},$$

and its extension

$$\tilde{\mathcal{H}} = \mathrm{span}_{\mathbf{C}}\{\alpha(m),\ c(2m),\ d(2m),\ c_0 |\ m \in \mathbf{Z}\},$$

with the Lie products

$$[\alpha(m),\ \alpha(n)] = (\alpha, \alpha) m \delta_{m+n,0} c_0,$$

$$[d(2m),\ c(2n)] = 2(d, c) m \delta_{m+n,0} c_0,$$

and

$$[\alpha(m), c(2n)] = [\alpha(m), d(2n)] = [c(2m), c(2n)] = [d(2m), d(2n)] = 0,$$

for $m, n \in \mathbf{Z}$.

For convenience, we set $c(2m+1) = 0 = d(2m+1)$ for $m \in \mathbf{Z}$, and define

$$\alpha_1(m) := \begin{cases} \alpha(m), & \text{if } m \in 2\mathbf{Z}+1, \\ 0, & \text{if } m \in 2\mathbf{Z}, \end{cases}$$

and $\alpha_2(m) := \alpha(m) - \alpha_1(m)$, for $m \in \mathbf{Z}$. We may extend the bilinear form $(\cdot,\cdot)$ to $\mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2 + \mathbf{Z}c$, by setting

$$(3.2.1) \qquad (\alpha_i, \alpha_j) = 4\delta_{ij}, \qquad (\alpha_i, c) = 0.$$

for $i, j = 1, 2$.

It is well known that $\mathcal{H}$ has an irreducible representation on the symmetric algebra $\mathcal{S}(\mathcal{H}^-)$, which is the polynomial algebra on variables $\alpha(k)$, $c(2k)$, and $d(2k)$ for $k \in \mathbf{Z}_-$, by setting c_0 to act as $\frac{1}{2}$, $a(-m)$ to act as multilication, and $a(m)$ to act as a partial differential operator for which

$$a(m).b(-n) = [a(m), b(-n)].1,$$

for $m, n \in \mathbf{Z}_+$, $a, b \in H$.

Let $\mathbf{C}[\Gamma]$ be the group algebra of the lattice Γ, with basis of the form e^γ, for $\gamma \in \Gamma$, and the multiplication $e^\gamma e^\mu = \epsilon(\gamma, \mu) e^{\gamma+\mu}$ for $\gamma, \mu \in \Gamma$, where $\epsilon: \ \Gamma \times \Gamma \longrightarrow \{\pm\}$ is the two-cocycle defined by setting

$$\epsilon(x, y) = \begin{cases} -1, & \text{if } (x, y) = (\alpha, c), \\ 1, & \text{otherwise}, \end{cases}$$

where $x, y \in \{\alpha, c, d\}$, and then extending to $\Gamma \times \Gamma$ by defining

$$\epsilon(\gamma + \mu, \lambda) = \epsilon(\gamma, \lambda)\epsilon(\mu, \lambda), \qquad \epsilon(\gamma, \mu + \lambda) = \epsilon(\gamma, \mu)\epsilon(\gamma, \lambda),$$

for $\gamma, \mu, \lambda \in \Gamma$. That is

$$(3.2.2) \qquad \epsilon(m_1\alpha + m_2 c + m_3 d, \ \ n_1\alpha + n_2 c + n_3 d) = (-1)^{m_1 n_2},$$

for $m_i, n_i \in \mathbf{Z}$.

Let ω_j, $j \in 2\mathbf{Z} + 1$ be symbols, we form a Clifford algebra $\mathcal{W}$ with the generators ω_j, $j \in 2\mathbf{Z} + 1$ satisfying the relations $\omega_i\omega_j + \omega_j\omega_i = -2\delta_{i+j,0}$ for $i, j \in 2\mathbf{Z} + 1$. We know that $\mathcal{W}$ has a standard irreducible representation on the exterior algebra $\Lambda(\mathcal{W}^-)$ on the generators ω_j, $j < 0$, with the action defined by

$$\omega_j.1 = 0, \qquad \omega_{-j}.\omega = \omega_{-j} \wedge \omega, \quad \text{for } j > 0,$$

which also implies

$$\omega_i.(\omega_{-j} \wedge \omega) = -2\delta_{i-j,0}\omega - \omega_{-j} \wedge (\omega_i.\omega).$$

for $i, j > 0$, $\omega \in \Lambda(\mathcal{W}^-)$.

Now we form the full Fock space.

$$V := \mathbb{C}[\Gamma] \otimes S(\mathcal{H}^-) \otimes \Lambda(\mathcal{W}^-). \tag{3.2.3}$$

and extend the actions of $\hat{\mathcal{H}}$, $\mathbb{C}[\Gamma]$ and $\mathcal{W}$ to V by defining

$$\begin{aligned} a(m).\epsilon^\gamma \otimes u \otimes \omega &= \epsilon^\gamma \otimes a(m).u \otimes \omega, \\ a(0).\epsilon^\gamma \otimes u \otimes \omega &= \frac{(a,\gamma)}{2}\epsilon^\gamma \otimes u \otimes \omega, \\ \epsilon^\mu.\epsilon^\gamma \otimes u \otimes \omega &= \epsilon(\mu,\gamma)\epsilon^{\mu+\gamma} \otimes u \otimes \omega, \\ \Omega.\epsilon^\gamma \otimes u \otimes \omega &= \epsilon^\gamma \otimes u \otimes \Omega.\omega, \end{aligned} \tag{3.2.4}$$

for $a \in H$, $m \in \mathbf{Z} \setminus \{0\}$, $\mu \in \Gamma$, $\Omega \in \mathcal{W}$ and $\epsilon^\gamma \otimes u \otimes \omega \in V$.

Lemma 3.2.1 $[\omega_i\omega_j, \ \omega_k] = 2\delta_{i+k,0}\omega_j - 2\delta_{j+k,0}\omega_i$, for $i, j, k \in 2\mathbf{Z} + 1$.

Proof. Since $\omega_i\omega_j + \omega_j\omega_i = -2\delta_{i+j,0}$, for $i, j \in 2\mathbf{Z} + 1$, we have

$$\begin{aligned} (\omega_i\omega_j)\omega_k &= \omega_i(-2\delta_{j+k,0} - \omega_k\omega_j) = -2\omega_i\delta_{j+k,0} - (-2\delta_{i+k,0} - \omega_k\omega_i)\omega_j \\ &= 2\delta_{i+k,0}\omega_j - 2\delta_{j+k,0}\omega_i + \omega_k(\omega_i\omega_j), \end{aligned}$$

as required.

□

Set $\omega_{2i} = 0$, for $i \in \mathbf{Z}$. We define an operator on $\Lambda(\mathcal{W}^-)$ (also on V) by

$$T_n = -\frac{1}{2}\sum_{i=1}^{\infty} i\omega_{\frac{n}{2}-i}\omega_{\frac{n}{2}+i}, \tag{3.2.5}$$

for $n \in 2\mathbf{Z}$. It is clear that T_n is an well-defined operator on $\Lambda(\mathcal{W}^-)$ (also on V).

Proposition 3.2.2 We have

(i). $[\omega_j, T_n] = (j + \frac{n}{2})\omega_{j+n}$,

(ii). $[T_m, T_n] = (m - n)T_{m+n} + \frac{m^3-4m}{24}\delta_{m+n,0}c_0$,

where $j \in 2\mathbf{Z} + 1$, $m, n \in 2\mathbf{Z}$.

Proof. We prove this proposition by applying the technique from [KR]. Let χ be a cut-off function on $\mathbf{R}$ defined by

$$\chi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

We put

$$T_n(\varepsilon) = -\frac{1}{2}\sum_{i=1}^{\infty} i\omega_{\frac{n}{2}-i}\omega_{\frac{n}{2}+i}\chi(\varepsilon i),$$

for $\varepsilon \in \mathbf{R}$. Note that $T_n(\varepsilon)$ is a finite summation over i if $\varepsilon \neq 0$, and $\lim_{\varepsilon\to 0} T_n(\varepsilon) = T_n$. The limit is understood to mean that, for any given $\omega \in \Lambda(\mathcal{W}^-)$ (or V), $T_n(\varepsilon).\omega = T_n.\omega$ provided ε sufficiently small. Now we prove (i). By the previous Lemma, we have

$$[\omega_j, T_n(\varepsilon)] = \frac{1}{2}\sum_{i=1}^{\infty} i[\omega_{\frac{n}{2}-i}\omega_{\frac{n}{2}+i}, \omega_j]\chi(i\varepsilon)$$

$$= \frac{1}{2}\sum_{i=1}^{\infty} i\{2\delta_{\frac{n}{2}-i+j,0}\omega_{\frac{n}{2}+i} - 2\delta_{\frac{n}{2}+i+j,0}\omega_{\frac{n}{2}-i}\}\chi(i\varepsilon)$$

$$= \begin{cases} (j+\frac{n}{2})\omega_{j+n}\chi((j+\frac{n}{2})\varepsilon), & \text{if } j+\frac{n}{2} \geq 0, \\ (j+\frac{n}{2})\omega_{j+n}\chi(-(j+\frac{n}{2})\varepsilon), & \text{if } j+\frac{n}{2} < 0. \end{cases}$$

This gives (i) by letting $\varepsilon \to 0$.

To show (ii), we need the following identity

$$\sum_{\substack{1\leq j\leq \frac{n}{2} \\ j \text{ odd}}} (j^2 + \frac{1}{4}n^2 - nj) = \frac{n^3-4n}{48}, \tag{3.2.6}$$

for $n \in 2\mathbf{Z}_+$, which is easy to check.

Now we prove (ii). By using (i), we have

$$[T_m(\varepsilon), T_n] = -\frac{1}{2}\sum_{i=1}^{\infty} i[\omega_{\frac{m}{2}-i}\omega_{\frac{m}{2}+i}, T_n]\chi(i\varepsilon)$$

$$= -\frac{1}{2}\sum_{i=1}^{\infty} i\left((\frac{m}{2}-i+\frac{n}{2})\omega_{\frac{m}{2}-i+n}\omega_{\frac{m}{2}+i} + (\frac{m}{2}+i+\frac{n}{2})\omega_{\frac{m}{2}-i}\omega_{\frac{m}{2}+i+n}\right)\chi(i\varepsilon)$$

$$= -\frac{1}{2}\sum_{j>-\frac{n}{2}} (j+\frac{n}{2})(\frac{m}{2}-j)\omega_{\frac{m+n}{2}-j}\omega_{\frac{m+n}{2}+j}\chi((j+\frac{n}{2})\varepsilon)$$

$$-\frac{1}{2}\sum_{j>\frac{n}{2}}(j-\frac{n}{2})(\frac{m}{2}+j)\omega_{\frac{m+n}{2}-j}\omega_{\frac{m+n}{2}+j}\chi((j-\frac{n}{2})\varepsilon).$$

where we have used the transformations $j=i-\frac{n}{2}$ for the first term. and $j=i+\frac{n}{2}$ for the second term. Now we split the above summations into four terms. and suppose $n\in 2\mathbf{Z}_+$. (likewise for the case $n\in -2\mathbf{N}$).

$$[T_m(\varepsilon).T_n]=\frac{1}{2}\sum_{j\geq 1}(j^2-\frac{mn}{4}-\frac{m-n}{2}j)\omega_{\frac{m+n}{2}-j}\omega_{\frac{m+n}{2}+j}\chi((j+\frac{n}{2})\varepsilon)$$

$$-\frac{1}{2}\sum_{j\geq 1}(j^2-\frac{mn}{4}+\frac{m-n}{2}j)\omega_{\frac{m+n}{2}-j}\omega_{\frac{m+n}{2}+j}\chi((j-\frac{n}{2})\varepsilon)$$

$$+\frac{1}{2}\sum_{j=-n/2}^{0}(j^2-\frac{mn}{4}-\frac{m-n}{2}j)\omega_{\frac{m+n}{2}-j}\omega_{\frac{m+n}{2}+j}\chi((j+\frac{n}{2})\varepsilon)$$

$$+\frac{1}{2}\sum_{j=1}^{n/2}(j^2-\frac{mn}{4}+\frac{m-n}{2}j)\omega_{\frac{m+n}{2}-j}\omega_{\frac{m+n}{2}+j}\chi((j-\frac{n}{2})\varepsilon).$$

We let $\varepsilon\to 0$. and obtain

$$[T_m.T_n]=(m-n)T_{m+n}$$

$$+\frac{1}{2}\sum_{j=1}^{n/2}(j^2-\frac{mn}{4}+\frac{m-n}{2}j)\{\omega_{\frac{m+n}{2}+j}\omega_{\frac{m+n}{2}-j}+\omega_{\frac{m+n}{2}-j}\omega_{\frac{m+n}{2}+j}\}-\frac{mn}{8}\omega_{\frac{m+n}{2}}\omega_{\frac{m+n}{2}}$$

$$=(m-n)T_{m+n}+\frac{1}{2}\sum_{\substack{1\leq j\leq n/2\\ j\text{ odd}}}(j^2-\frac{mn}{4}+\frac{m-n}{2}j)\{-2\delta_{m+n,0}\}$$

$$=(m-n)T_{m+n}-\sum_{\substack{1\leq j\leq n/2\\ j\text{ odd}}}(j^2+\frac{n^2}{4}-nj)\delta_{m+n,0}$$

$$=(m-n)T_{m+n}+\frac{m^3-4m}{24}\delta_{m+n,0}c_0.$$

as required.

□

Recall that $\alpha(m)=\alpha_1(m)+\alpha_2(m)$ for $m\in\mathbf{Z}$. We define. for $\beta\in\mathbf{Z}\alpha_1+\mathbf{Z}\alpha_2+\mathbf{Z}c$

$$\beta(z)=\sum_{j\in\mathbf{Z}}\beta(j)z^{-j}.\qquad W(z)=\sum_{j\in 2\mathbf{Z}+1}\omega_j z^{-j}.$$

and

$$E^{\pm}(\beta, z) = \exp(-\sum_{k\in \mathbf{Z}_{\pm}} \frac{\beta(k)}{k} z^{-k}).$$

where exp denotes the formal exponential power series. When $E^{\pm}(\beta, z)$ is expanded into a formal Laurent series in z, the coefficients of z^{-j} ($j \in \mathbf{Z}$) are operators on V.

Lemma 3.2.3 Let z_1, z_2 be formal variables, we have

$$W(z_1)W(z_2) + W(z_2)W(z_1) = \delta(-\frac{z_2}{z_1}) - \delta(\frac{z_2}{z_1}).$$

Proof. we have

$$W(z_1)W(z_2) + W(z_2)W(z_1)$$

$$= \sum_{i,j\in 2\mathbf{Z}+1} \omega_i\omega_j z_1^{-i} z_2^{-j} + \sum_{i,j\in 2\mathbf{Z}+1} \omega_j\omega_i z_1^{-i} z_2^{-j}$$

$$= \sum_{i,j\in 2\mathbf{Z}+1} (\omega_i\omega_j + \omega_j\omega_i) z_1^{-i} z_2^{-j} = \sum_{i,j\in 2\mathbf{Z}+1} (-2\delta_{i+j,0}) z_1^{-i} z_2^{-j}$$

$$= -2 \sum_{i\in 2\mathbf{Z}+1} (\frac{z_2}{z_1})^i = -(\delta(\frac{z_2}{z_1}) - \delta(-\frac{z_2}{z_1})),$$

as required.

□

In addition to the formal exponential series notation $\exp(x)$,we shall use the following formal power series as we did in Chapter 1.

$$\log(1 + ax) = -\sum_{k=1}^{\infty} \frac{(-a)^k}{k} x^k, \qquad (1 + x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k, \tag{3.2.7}$$

for $a \in \mathbf{C}$, and formal variable x, where the symbol $\binom{a}{k}$ is defined by

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!},$$

for $a \in \mathbf{C}$, $k \in \mathbf{N}$.

In fact, the above expressions obey the following standard rules, see [FLM]

$$\log(\exp(x)) = x, \qquad \exp(\log(1 + ax)) = 1 + ax, \tag{3.2.8}$$

$$\log((1 + ax)(1 + bx)) = \log(1 + ax) + \log(1 + bx), \qquad \log(1 + ax)^b = b\log(1 + ax),$$

for $a, b \in \mathbb{C}$, and formal variable x.

In particular, we have

$$(3.2.9) \qquad (1-x)^{-1} = \sum_{k=0}^{\infty} x^k, \qquad (1-x)^{-2} = \sum_{k=0}^{\infty}(k+1)x^k,$$

$$(1+x)(1-x)^{-1} = 1 + 2\sum_{k=1}^{\infty} x^k.$$

Lemma 3.2.4 We have

$$W(z_1)W(z_2)\frac{z_2}{z_1}(1-\frac{z_2^2}{z_1^2})^{-1} - W(z_2)W(z_1)\frac{z_1}{z_2}(1-\frac{z_1^2}{z_2^2})^{-1}$$

$$= -\frac{1}{2}((D\delta)(\frac{z_2}{z_1}) + (D\delta)(-\frac{z_2}{z_1})).$$

Proof. By (3.2.9), we have

$$I := W(z_1)W(z_2)\frac{z_2}{z_1}(1-\frac{z_2^2}{z_1^2})^{-1} - W(z_2)W(z_1)\frac{z_1}{z_2}(1-\frac{z_1^2}{z_2^2})^{-1}$$

$$= W(z_1)W(z_2)\sum_{k=0}^{\infty}(\frac{z_2}{z_1})^{2k+1} - W(z_2)W(z_1)\sum_{k=0}^{\infty}(\frac{z_1}{z_2})^{2k+1}$$

$$= \sum_{i,j\in 2\mathbf{Z}+1}\sum_{k=0}^{\infty}\omega_i\omega_j z_1^{-i-2k-1}z_2^{-j+2k+1} - \sum_{i,j\in 2\mathbf{Z}+1}\sum_{k=0}^{\infty}\omega_j\omega_i z_1^{-i+2k+1}z_2^{-j-2k-1}$$

$$= \sum_{i,j\in 2\mathbf{Z}} a_{ij} z_1^{-i} z_2^{-j},$$

where

$$a_{ij} = \sum_{k=0}^{\infty}(\omega_{i-2k-1}\omega_{j+2k+1} - \omega_{j-2k-1}\omega_{i+2k+1})$$

$$= \begin{cases} \omega_{i-1}\omega_{j+1} + \omega_{i-3}\omega_{j+3} + \cdots + \omega_{j+1}\omega_{i-1}, & \text{if } i > j, \\ 0, & \text{if } i = j, \\ -\{\omega_{j-1}\omega_{i+1} + \omega_{j-3}\omega_{i+3} + \cdots + \omega_{i+1}\omega_{j-1}\}, & \text{if } i < j. \end{cases}$$

In all cases, we have $a_{ij} = -i\delta_{i+j,0}$. Therefore, we obtain

$$I = \sum_{i,j\in 2\mathbf{Z}}(-i\delta_{i+j,0})z_1^{-i}z_2^{-j} = -\sum_{i\in 2\mathbf{Z}} i(\frac{z_2}{z_1})^i$$

$$= -\frac{1}{2}((D\delta)(\frac{z_2}{z_1}) + (D\delta)(-\frac{z_2}{z_1})).$$

as required.

$\square$

Lemma 3.2.5 Let $\beta_1, \beta_2 \in \mathbf{Z}\alpha + \mathbf{Z}c$, we have

$$[\beta_1(z_1), \beta_2(z_2)] = \frac{1}{2}(\beta_1, \beta_2)(D\delta)(\frac{z_2}{z_1}).$$

Proof. we have

$$\begin{aligned}
[\beta_1(z_1), \beta_2(z_2)] &= \sum_{i,j \in \mathbf{Z}} z_1^{-i} z_2^{-j} [\beta_1(i), \beta_2(j)] \\
&= \sum_{i,j \in \mathbf{Z}} z_1^{-i} z_2^{-j} (\beta_1, \beta_2) i \delta_{i+j,0} c_0 \\
&= \frac{(\beta_1, \beta_2)}{2} \sum_{i \in \mathbf{Z}} i (\frac{z_2}{z_1})^i = \frac{(\beta_1, \beta_2)}{2} (D\delta)(\frac{z_2}{z_1}).
\end{aligned}$$

as required.

$\square$

Lemma 3.2.6 Let $\beta = \beta_1 + \beta_2$, $\tau = \tau_1 + \tau_2$, for $\beta_1, \tau_1 \in \mathbf{Z}\alpha_1 + \mathbf{Z}c$, and $\beta_2, \tau_2 \in \mathbf{Z}\alpha_2 + \mathbf{Z}c$. We have

$$(i). \quad [\beta(z_1), E^-(\tau, z_2)] = E^-(\tau, z_2) \left(\frac{(\beta_1, \tau_1)}{2} \sum_{k \in 2\mathbf{N}+1} (\frac{z_2}{z_1})^k + \frac{(\beta_2, \tau_2)}{2} \sum_{k \in 2\mathbf{Z}_+} (\frac{z_2}{z_1})^k \right).$$

$$(ii). \quad [\beta(z_1), E^+(\tau, z_2)] = E^+(\tau, z_2) \left(\frac{(\beta_1, \tau_1)}{2} \sum_{k \in 2\mathbf{N}+1} (\frac{z_1}{z_2})^k + \frac{(\beta_2, \tau_2)}{2} \sum_{k \in 2\mathbf{Z}_+} (\frac{z_1}{z_2})^k \right).$$

$$(iii). \quad E^+(\beta, z_1) E^-(\tau, z_2) = E^-(\tau, z_2) E^+(\beta, z_1) (1 + \frac{z_2}{z_1})^{\frac{(\beta_2, \tau_2) - (\beta_1, \tau_1)}{4}} (1 - \frac{z_2}{z_1})^{\frac{(\beta_2, \tau_2) + (\beta_1, \tau_1)}{4}}.$$

Proof. Recall the formal rules [cf. [FLM]], if $[X, Y]$ commutes with X and Y,

$$[X, e^Y] = e^Y [X, Y]. \tag{3.2.10}$$

and

$$e^X e^Y = e^Y e^X e^{[X,Y]}. \tag{3.2.11}$$

and note that

$$[\beta(z_1), \sum_{k=1}^{\infty} \frac{\tau(-k)}{k} z_2^k]$$

$$= [\beta_1(z_1) + \beta_2(z_1), \sum_{k=1}^{\infty} \frac{\tau_1(-k) + \tau_2(-k)}{k} z_2^k]$$

$$= \sum_{i \in 2\mathbf{Z}+1} \sum_{k \in 2\mathbf{N}+1} \frac{1}{k} [\beta_1(i), \tau_1(-k)] z_1^{-i} z_2^k$$

$$+ \sum_{i \in 2\mathbf{Z}} \sum_{k \in 2\mathbf{Z}_+} \frac{1}{k} [\beta_2(i), \tau_2(-k)] z_1^{-i} z_2^k$$

$$= \sum_{i \in 2\mathbf{Z}+1} \sum_{k \in 2\mathbf{N}+1} \frac{1}{k} (\beta_1, \tau_1) i \delta_{i-k,0} \frac{1}{2} z_1^{-i} z_2^k$$

$$+ \sum_{i \in 2\mathbf{Z}} \sum_{k \in 2\mathbf{Z}_+} \frac{1}{k} (\beta_2, \tau_2) i \delta_{i-k,0} \frac{1}{2} z_1^{-i} z_2^k$$

$$= \frac{(\beta_1, \tau_1)}{2} \sum_{k \in 2\mathbf{N}+1} (\frac{z_2}{z_1})^k + \frac{(\beta_2, \tau_2)}{2} \sum_{k \in 2\mathbf{Z}_+} (\frac{z_2}{z_1})^k.$$

This and (3.2.10) give the identity (i). Likewise, we can prove (ii). Now, for (iii), we compute

$$[-\sum_{k=1}^{\infty} \frac{\beta(k)}{k} z_1^{-k}, \sum_{l=1}^{\infty} \frac{\tau(-l)}{l} z_2^l]$$

$$= - \sum_{k,l \in 2\mathbf{N}+1} \frac{1}{kl} [\beta_1(k), \tau_1(-l)] z_1^{-k} z_2^l$$

$$- \sum_{k,l \in 2\mathbf{Z}_+} \frac{1}{kl} [\beta_2(k), \tau_2(-l)] z_1^{-k} z_2^l$$

$$= - \sum_{k,l \in 2\mathbf{N}+1} \frac{1}{kl} k (\beta_1, \tau_1) \delta_{k-l,0} \frac{1}{2} z_1^{-k} z_2^l$$

$$- \sum_{k,l \in 2\mathbf{Z}_+} \frac{1}{kl} k (\beta_2, \tau_2) \delta_{k-l,0} \frac{1}{2} z_1^{-k} z_2^l$$

$$= -\frac{(\beta_1, \tau_1)}{2} \sum_{k \in 2\mathbf{N}+1} \frac{1}{k} (\frac{z_2}{z_1})^k - \frac{(\beta_2, \tau_2)}{2} \sum_{k \in 2\mathbf{Z}_+} \frac{1}{k} (\frac{z_2}{z_1})^k.$$

$$= -\frac{(\beta_1, \tau_1)}{4} \log\left((1 + \frac{z_2}{z_1})(1 - \frac{z_2}{z_1})^{-1}\right) + \frac{(\beta_2, \tau_2)}{4} \log(1 - \frac{z_2^2}{z_1^2})$$

$$= \log\{(1 + \frac{z_2}{z_1})^{\frac{(\beta_2, \tau_2) - (\beta_1, \tau_1)}{4}} (1 - \frac{z_2}{z_1})^{\frac{(\beta_2, \tau_2) + (\beta_1, \tau_1)}{4}}\}.$$

This and (3.2.11) give the identity (iii).

$\square$

Now we define the vertex operators which correspond to the fields $C_i(z, 2n)$, $\alpha(z, n)$ and $x_\pm(z, n)$ of the Lie algebra $\hat{\mathcal{G}}(\mathcal{T})$, for $n \in \mathbf{Z}$, $i = 1, 2$, [see Proposition 2.5.3].

(3.2.12) $$C_1(z, 2n) = \frac{1}{2}\epsilon^{2nc} z^{2nc} E^-(2nc, z) E^+(2nc, z),$$

$$C_2(z, 2n) = \frac{1}{2}\epsilon^{2nc} z^{2nc} E^-(2nc, z) c(z) E^+(2nc, z),$$

$$\alpha(z, n) = \begin{cases} \epsilon^{nc} z^{nc} E^-(nc, z)\alpha(z) E^+(nc, z), & \text{if } n \in 2\mathbf{Z}, \\ -\epsilon^{nc} z^{nc} E^-(nc + \alpha_1, z) W(z) E^+(nc + \alpha_1, z), & \text{if } n \in 2\mathbf{Z} + 1, \end{cases}$$

$$x_\pm(z, n) = \begin{cases} z\epsilon^{nc\pm\alpha} z^{nc\pm\alpha} E^-(nc \pm \alpha, z) E^+(nc \pm \alpha, z), & \text{if } n \in 2\mathbf{Z}, \\ z\epsilon^{nc\pm\alpha} z^{nc\pm\alpha} E^-(nc \pm \alpha_2, z) W(z) E^+(nc \pm \alpha_2, z), & \text{if } n \in 2\mathbf{Z} + 1, \end{cases}$$

for $n \in \mathbf{Z}$, where the operator z^β is defined by

(3.2.13) $$z^\beta . \epsilon^\gamma \otimes u \otimes \omega = z^{\frac{(\beta, \gamma)}{2}} \epsilon^\gamma \otimes u \otimes \omega,$$

for $\beta \in \mathbf{Z}\alpha + \mathbf{Z}c$, $\epsilon^\gamma \otimes u \otimes \omega \in V$.

Strictly speaking, the vertex operators, given in (3.2.12), are not operators on V, but the vertex operators can be formally expanded into power series in z, and the coefficients of z^i $(i \in \mathbf{Z})$ are indeed operators acting on V.

Lemma 3.2.7 Let $\beta_1, \beta_2 \in \mathbf{Z}\alpha + \mathbf{Z}c$. Then

$$(i). \quad z^{\beta_1} \epsilon^{\beta_2} = z^{\frac{(\beta_1, \beta_2)}{2}} \epsilon^{\beta_2} z^{\beta_1}.$$

$$(ii). \quad [\beta_1(z), \epsilon^{\beta_2}] = \frac{1}{2}(\beta_1, \beta_2)\epsilon^{\beta_2}.$$

Proof. Let $\epsilon^\gamma \otimes u \otimes \omega \in V$, we have

$$z^{\beta_1} \epsilon^{\beta_2} . \epsilon^\gamma \otimes u \otimes \omega = \epsilon(\beta_2, \gamma) z^{\beta_1} . \epsilon^{\beta_2 + \gamma} \otimes u \otimes \omega$$

$$= \epsilon(\beta_2, \gamma) z^{\frac{(\beta_1, \beta_2 + \gamma)}{2}} \epsilon^{\beta_2 + \gamma} \otimes u \otimes \omega$$

$$= z^{\frac{(\beta_1, \beta_2)}{2}} \epsilon^{\beta_2} z^{\beta_1} . \epsilon^\gamma \otimes u \otimes \omega.$$

This gives (i). Similarly, for (ii), we have

$$[\beta_1(z), \epsilon^{\beta_2}] . \epsilon^\gamma \otimes u \otimes \omega = [\beta_1(0), \epsilon^{\beta_2}] . \epsilon^\gamma \otimes u \otimes \omega$$

$$= \left(\epsilon(\beta_2,\gamma)\frac{(\beta_1,\beta_2+\gamma)}{2} - \frac{(\beta_1,\gamma)}{2}\epsilon(\beta_2,\gamma)\right)\epsilon^{\gamma}\otimes u\otimes\omega$$

$$= \frac{(\beta_1,\beta_2)}{2}\epsilon^{\beta_2}.\epsilon^{\gamma}\otimes u\otimes\omega.$$

as required.

□

To close this section, we state the main result of this chapter.

Theorem 3.2.8 The vertex operators given by (3.2.12) satisfy the commutation relations (2.5.2)-(2.5.14). That is, the coefficient operators, acting on $V = \mathbb{C}[\Gamma]\otimes S(\mathcal{H}^-)\otimes\Lambda(\mathcal{W}^-)$, of the vertex operators (3.2.12) span a Lie algebra which is isomorphic to the universal central extension $\hat{\mathcal{G}}(\mathcal{T})$ of the TKK algebra $\mathcal{G}(\mathcal{T})$.

§3.3 Proof of Theorem 3.2.8

To proof Theorem 3.2.8, we are required to check that the vertex operators given by (3.2.12) satisfy all commutation relations (2.5.2) to (2.5.14) given in Proposition 2.5.3. In the proof, we will freely use Lemma 1.5.1 of Chapter 1. Now, we will check the relations one by one in the following order (2.5.11), (2.5.12), (2.5.13), (2.5.7), (2.5.8), (2.5.9), (2.5.10), (2.5.3), (2.5.4), (2.5.5), (2.5.6), (2.5.14) and (2.5.2).

For (2.5.11). Applying Lemma 3.2.5, and Lemma 1.5.1, we have

$$[\alpha(z_1,2m),\alpha(z_2,2n)]$$

$$= [\epsilon^{2mc}z_1^{2mc}E^-(2mc,z_1)\alpha(z_1)E^+(2mc,z_1),\epsilon^{2nc}z_2^{2nc}E^-(2nc,z_2)\alpha(z_2)E^+(2nc,z_2)]$$

$$= \epsilon^{2(m+n)c}z_1^{2mc}z_2^{2nc}E^-(2mc,z_1)E^-(2nc,z_2)[\alpha(z_1),\alpha(z_2)]E^+(2mc,z_1)E^+(2nc,z_2)$$

$$= 2\epsilon^{2(m+n)c}z_1^{2mc}z_2^{2nc}E^-(2mc,z_1)E^-(2nc,z_2)E^+(2mc,z_1)E^+(2nc,z_2)(D\delta)(\frac{z_2}{z_1})$$

$$= 2\epsilon^{2(m+n)c}z_2^{2(m+n)c}E^-(2(m+n)c,z_2)E^+(2(m+n)c,z_2)(D\delta)(\frac{z_2}{z_1})$$

$$+2\epsilon^{2(m+n)c}z_2^{2(m+n)c}E^-(2(m+n)c,z_2)\left(\sum_{k\in 2\mathbb{Z}}2mc(-k)z_2^k\right)E^+(2(m+n)c,z_2)\delta(\frac{z_2}{z_1})$$

$$= 4C_1(z_2, 2(m+n))(D\delta)(\frac{z_2}{z_1}) + 8mC_2(z_2, 2(m+n))\delta(\frac{z_2}{z_1}),$$

as required.

For (2.5.12). Applying Lemma 3.2.6 and Lemma 1.5.1, we have

$$[\alpha(z_1, 2m), \alpha(z_2, 2n-1)]$$

$$= [\epsilon^{2mc} z_1^{2mc} E^-(2mc, z_1)\alpha(z_1)E^+(2mc, z_1), -\epsilon^{(2n-1)c} z_2^{(2n-1)c} E^-((2n-1)c + \alpha_1, z_2)$$

$$\cdot W(z_2)E^+((2n-1)c + \alpha_1, z_2)]$$

$$= -\epsilon^{(2m+2n-1)c} z_1^{2mc} z_2^{(2n-1)c} E^-(2mc, z_1)\{[\alpha(z_1), E^-((2n-1)c+\alpha_1, z_2)]W(z_2)E^+((2n-1)c+\alpha_1, z_2)$$

$$+E^-((2n-1)c + \alpha_1, z_2)W(z_2)[\alpha(z_1), E^+((2n-1)c + \alpha_1, z_2)]\}E^+(2mc, z_1)$$

$$= -\epsilon^{(2m+2n-1)c} z_1^{2mc} z_2^{(2n-1)c} E^-(2mc, z_1)E^-((2n-1)c + \alpha_1, z_2)W(z_2)E^+(2mc, z_1)$$

$$\cdot E^+((2n-1)c + \alpha_1, z_2)\left(2\sum_{k\in 2\mathbf{N}+1}(\frac{z_2}{z_1})^k + 2\sum_{k\in 2\mathbf{N}+1}(\frac{z_1}{z_2})^k\right)$$

$$= \alpha(z_2, 2m+2n-1)(\delta(\frac{z_2}{z_1}) - \delta(-\frac{z_2}{z_1})),$$

as required.

For (2.5.13). By Lemma 3.2.6, we have

$$\alpha(z_1, 2m+1)\alpha(z_2, 2n-1)$$

$$= \epsilon^{(2m+1)c} z_1^{(2m+1)c} E^-((2m+1)c + \alpha_1, z_1)W(z_1)E^+((2m+1)c + \alpha_1, z_1)$$

$$\cdot\epsilon^{(2n-1)c} z_2^{(2n-1)c} E^-((2n-1)c + \alpha_1, z_2)W(z_2)E^+((2n-1)c + \alpha_1, z_2)$$

$$= \epsilon^{(2m+2n)c} z_1^{(2m+1)c} z_2^{(2n-1)c} E^-((2m+1)c + \alpha_1, z_1)E^-((2n-1)c + \alpha_1, z_2)$$

$$\cdot\{W(z_1)W(z_2)(1+\frac{z_2}{z_1})^{-1}(1-\frac{z_2}{z_1})\}E^+((2m+1)c + \alpha_1, z_1)E^+((2n-1)c + \alpha_1, z_2).$$

From this, we obtain

$$[\alpha(z_1, 2m+1), \alpha(z_2, 2n-1)]$$

$$= \epsilon^{(2m+2n)c} z_1^{(2m+1)c} z_2^{(2n-1)c} E^-((2m+1)c + \alpha_1, z_1)E^-((2n-1)c + \alpha_1, z_2)$$

$$\cdot P(z_1, z_2)E^+((2m+1)c + \alpha_1, z_1)E^+((2n-1)c + \alpha_1, z_2).$$

where

$$P(z_1, z_2) = W(z_1)W(z_2)(1+\frac{z_2}{z_1})^{-1}(1-\frac{z_2}{z_1})\} - W(z_2)W(z_1)(1+\frac{z_1}{z_2})^{-1}(1-\frac{z_1}{z_2}).$$

Therefore, by Lemma 1.8.1, we obtain

$$[\alpha(z_1, 2m+1), \alpha(z_2, 2n-1)]$$

$$= 2\epsilon^{(2m+2n)c} z_1^{(2m+1)c} z_2^{(2n-1)c} E^-((2m+1)c+\alpha_1, z_1)E^-((2n-1)c+\alpha_1, z_2)$$
$$\cdot E^+((2m+1)c+\alpha_1, z_1)E^+((2n-1)c+\alpha_1, z_2)(D\delta)(-\frac{z_2}{z_1}).$$

By Lemma 1.5.1, and note that

$$E^{\pm}((2m+1)c+\alpha_1, z_1)E^{\pm}((2n-1)c+\alpha_1, z_2)|_{z_1=-z_2}$$
$$= E^{\pm}((2m+2n)c, z_2),$$

we have

$$[\alpha(z_1, 2m+1), \alpha(z_2, 2n-1)]$$
$$= 2\epsilon^{(2m+2n)c} z_2^{(2m+2n)c} E^-((2m+2n)c, z_2)E^+((2m+2n)c, z_2)(D\delta)(-\frac{z_2}{z_1})$$
$$+2\epsilon^{(2m+2n)c} z_2^{(2m+2n)c} E^-((2m+2n)c, z_2)\{\alpha_1(-z_2)+(2m+1)c(-z_2)\}$$
$$\cdot E^+((2m+2n)c, z_2)\delta(-\frac{z_2}{z_1}),$$

where $c(-z_2) = c(z_2)$, and

$$\alpha_1(-z_2) = -(\alpha(z_2, 2m+2n) - \alpha(-z_2, 2m+2n)).$$

Therefore, we obtain

$$[\alpha(z_1, 2m+1), \alpha(z_2, 2n-1)]$$
$$= 4C_1(z_2, 2m+2n)(D\delta)(-\frac{z_2}{z_1}) + 4(2m+1)C_2(z_2, 2m+2n)\delta(-\frac{z_2}{z_1})$$
$$-(\alpha(z_2, 2m+2n) - \alpha(-z_2, 2m+2n))\delta(-\frac{z_2}{z_1}),$$

as required.

For (2.5.7). By Lemma 3.2.6, Lemma 3.2.7 and Lemma 1.5.1, we have

$$[\alpha(z_1, 2m), x_\pm(z_2, 2n)]$$

$$= [\epsilon^{2mc} z_1^{2mc} E^-(2mc, z_1)\alpha(z_1)E^+(2mc, z_1), z_2\epsilon^{2nc\pm\alpha} z_2^{2nc\pm\alpha} E^-(2nc \pm \alpha, z_2)$$

$$\cdot E^+(2nc + \alpha, z_2)]$$

$$= z_2\epsilon^{2mc} z_1^{2mc} E^-(2mc, z_1)[\alpha(z_1), \epsilon^{2nc\pm\alpha} z_2^{2nc\pm\alpha} E^-(2nc \pm \alpha, z_2)E^+(2nc \pm \alpha, z_2)]$$

$$\cdot E^+(2mc, z_1)$$

$$= z_2\epsilon^{2mc} z_1^{2mc} E^-(2mc, z_1)\{\frac{1}{2}(\alpha, \pm\alpha) + \frac{1}{2}(\alpha_1, \pm\alpha_1) \sum_{k\in 2\mathbf{N}+1} (\frac{z_2}{z_1})^k$$

$$+\frac{1}{2}(\alpha_2, \pm\alpha_2) \sum_{k\in 2\mathbf{Z}_+} (\frac{z_2}{z_1})^k + \frac{1}{2}(\alpha_1, \pm\alpha_1) \sum_{k\in 2\mathbf{N}+1} (\frac{z_1}{z_2})^k + \frac{1}{2}(\alpha_2, \pm\alpha_2) \sum_{k\in 2\mathbf{Z}_+} (\frac{z_1}{z_2})^k\}$$

$$\cdot\epsilon^{2nc\pm\alpha} z_2^{2nc\pm\alpha} E^-(2nc \pm \alpha, z_2)E^+(2nc \pm \alpha, z_2)E^+(2mc, z_1)$$

$$= \pm 2z_2\epsilon^{2mc+2nc\pm\alpha} z_1^{2mc} z_2^{2nc\pm\alpha} E^-(2mc, z_1)E^-(2nc \pm \alpha, z_2)$$

$$\cdot E^+(2mc, z_1)E^+(2nc \pm \alpha, z_2)\delta(\frac{z_2}{z_1})$$

$$= \pm 2x_\pm(z_2, 2m + 2n)\delta(\frac{z_2}{z_1}),$$

as required.

For (2.5.8). Similarly as in the proof (2.5.7), we have

$$[\alpha(z_1, 2m), x_\pm(z_2, 2n - 1)]$$

$$= [\epsilon^{2mc} z_1^{2mc} E^-(2mc, z_1)\alpha(z_1)E^+(2mc, z_1), x_\pm(z_2, 2n - 1)]$$

$$= \epsilon^{2mc} z_1^{2mc} E^-(2mc, z_1)[\alpha(z_1), x_\pm(z_2, 2n - 1)]E^+(2mc, z_1),$$

where

$$[\alpha(z_1), x_\pm(z_2, 2n - 1)] =$$

$$= [\alpha(z_1), z_2\epsilon^{(2n-1)c\pm\alpha} z_2^{(2n-1)c\pm\alpha} E^-((2n - 1)c \pm \alpha_2, z_2)W(z_2)E^+((2n - 1)c + \alpha_2, z_2)]$$

$$= z_2\epsilon^{(2n-1)c\pm\alpha} z_2^{(2n-1)c\pm\alpha} E^-((2n - 1)c \pm \alpha_2, z_2)W(z_2)E^+((2n - 1)c + \alpha_2, z_2)$$

$$\cdot\left(\pm 2 \pm 2 \sum_{k\in 2\mathbb{Z}_+} (\frac{z_2}{z_1})^k \pm 2 \sum_{k\in 2\mathbb{Z}_+} (\frac{z_1}{z_2})^k\right)$$

$$= \pm z_2 \epsilon^{(2n-1)c\pm\alpha} z_2^{(2n-1)c\pm\alpha} E^-((2n-1)c \pm \alpha_2, z_2) W(z_2) E^+((2n-1)c + \alpha_2, z_2)$$

$$\cdot(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1})).$$

Therefore, we obtain

$$[\alpha(2mc, z_1), x_\pm(z_2, 2n-1)]$$

$$= \pm z_2 \epsilon^{(2m+2n-1)c\pm\alpha} z_1^{2mc} z_2^{(2n-1)c\pm\alpha} E^-(2mc, z_1) E^-((2n-1)c \pm \alpha_2, z_2) W(z_2)$$

$$\cdot E^+(2mc, z_1) E^+((2n-1)c + \alpha_2, z_2)(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1}))$$

$$= \pm x_\pm(z_2, 2m+2n-1)(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1})),$$

as required.

For (2.5.9). By (3.2.2) and Lemma 3.2.6, we have

$$\alpha(z_1, 2m+1) x_\pm(z_2, 2n)$$

$$= -\epsilon^{(2m+1)c} z_1^{(2m+1)c} E^-((2m+1)c + \alpha_1, z_1) W(z_1) E^+((2m+1)c + \alpha_1, z_1)$$

$$\cdot z_2 \epsilon^{2nc\pm\alpha} z_2^{2nc\pm\alpha} E^-(2nc \pm \alpha, z_2) E^+(2nc \pm \alpha, z_2)$$

$$= -z_2 \epsilon^{(2m+2n+1)c\pm\alpha} z_1^{(2m+1)c} z_2^{2nc\pm\alpha} E^-((2m+1)c + \alpha_1, z_1) E^-(2nc \pm \alpha, z_2) W(z_1)$$

$$\cdot(1 + \frac{z_2}{z_1})^{\mp 1}(1 - \frac{z_2}{z_1})^{\pm 1} E^+((2m+1)c + \alpha_1, z_1) E^+(2nc \pm \alpha, z_2).$$

We also have

$$x_\pm(z_2, 2n)\alpha(z_1, 2m+1)$$

$$= -z_2 \epsilon^{2nc\pm\alpha} z_2^{2nc\pm\alpha} E^-(2nc \pm \alpha, z_2) E^+(2nc \pm \alpha, z_2)$$

$$\cdot \epsilon^{(2m+1)c} z_1^{(2m+1)c} E^-((2m+1)c + \alpha_1, z_1) W(z_1) E^+((2m+1)c + \alpha_1, z_1)$$

$$= -z_2 \epsilon(2nc \pm \alpha, (2m+1)c) \epsilon^{(2m+2n+1)c\pm\alpha} z_1^{(2m+1)c} z_2^{2nc\pm\alpha} E^-((2m+1)c + \alpha_1, z_1)$$

$$\cdot E^-(2nc \pm \alpha, z_2) W(z_1) E^+((2m+1)c + \alpha_1, z_1)$$

$$\cdot E^+(2nc \pm \alpha, z_2)(1 + \frac{z_1}{z_2})^{\mp 1}(1 - \frac{z_1}{z_2})^{\pm 1}.$$

Therefore, we obtain

$$[\alpha(z_1, 2m+1), x_\pm(z_2, 2n)]$$

$$= -z_2\epsilon^{(2m+2n+1)c\pm\alpha} z_1^{(2m+1)c} z_2^{2nc\pm\alpha} E^-((2m+1)c+\alpha_1, z_1)E^-(2nc\pm\alpha, z_2)$$

$$\cdot W(z_1)E^+((2m+1)c+\alpha_1, z_1)E^+(2nc\pm\alpha, z_2)Q(z_1, z_2),$$

where

$$Q(z_1, z_2) = (1+\frac{z_2}{z_1})^{\mp 1}(1-\frac{z_2}{z_1})^{\pm 1} + (1+\frac{z_1}{z_2})^{\mp 1}(1-\frac{z_1}{z_2})^{\pm 1}$$

$$= 1 + 2\sum_{k=1}^{\infty}(\mp\frac{z_2}{z_1})^k + 1 + 2\sum_{k=1}^{\infty}(\mp\frac{z_1}{z_2})^k = 2\delta(\mp\frac{z_2}{z_1}).$$

Thus,

$$[\alpha(z_1, 2m+1), x_\pm(z_2, 2n)]$$

$$= -2z_2\epsilon^{(2m+2n+1)c\pm\alpha} z_1^{(2m+1)c} z_2^{2nc\pm\alpha} E^-((2m+1)c+\alpha_1, z_1)E^-(2nc\pm\alpha, z_2)$$

$$\cdot W(z_1)E^+((2m+1)c+\alpha_1, z_1)E^+(2nc\pm\alpha, z_2)\delta(\mp\frac{z_2}{z_1})$$

$$= \pm 2z_2\epsilon^{(2m+2n+1)c\pm\alpha} z_2^{(2m+2n+1)c\pm\alpha} E^-((2m+2n+1)c\pm\alpha_2, z_2)W(z_2)$$

$$\cdot E^+((2m+2n+1)c\pm\alpha_2, z_2)\delta(\mp\frac{z_2}{z_1})$$

$$= \pm 2x_\pm(z_2, 2m+2n+1)\delta(\mp\frac{z_2}{z_1}),$$

as required.

For (2.5.10). We have

$$[\alpha(z_1, 2m+1), x_\pm(z_2, 2n-1)]$$

$$= [-\epsilon^{(2m+1)c} z_1^{(2m+1)c} E^-((2m+1)c+\alpha_1, z_1)W(z_1)E^+((2m+1)c+\alpha_1, z_1),$$

$$z_2\epsilon^{(2n-1)c\pm\alpha} z_2^{(2n-1)c\pm\alpha} E^-((2n-1)c\pm\alpha_2, z_2)W(z_2)E^+((2n-1)c\pm\alpha_2, z_2)]$$

$$= -z_2\epsilon^{(2m+2n)c\pm\alpha} z_1^{(2m+1)c} z_2^{(2n-1)c\pm\alpha} E^-((2m+1)c+\alpha_1, z_1)E^-((2n-1)c\pm\alpha_2, z_2)$$

$$\cdot R(z_1, z_2)E^+((2m+1)c+\alpha_1, z_1)E^+((2n-1)c\pm\alpha_2, z_2),$$

where, By Lemma 3.2.3

$$R(z_1, z_2) =$$

$$\epsilon((2m+1)c,(2n-1)c\pm\alpha)W(z_1)W(z_2)-\epsilon((2n-1)c\pm\alpha,(2m+1)c)W(z_2)W(z_1)$$
$$=W(z_1)W(z_2)+W(z_2)W(z_1)=\delta(-\frac{z_2}{z_1})-\delta(\frac{z_2}{z_1}).$$

Therefore, we have

$$[\alpha(z_1,2m+1),x_\pm(z_2,2n-1)]$$
$$=z_2\epsilon^{(2m+2n)c\pm\alpha}z_2^{(2m+2n)c\pm\alpha}E^-((2m+1)c+\alpha_1,z_2)E^-((2n-1)c\pm\alpha_2,z_2)$$
$$E^+((2m+1)c+\alpha_1,z_2)E^+((2n-1)c\pm\alpha_2,z_2)\delta(\frac{z_2}{z_1})$$
$$-z_2\epsilon^{(2m+2n)c\pm\alpha}z_2^{(2m+2n)c\pm\alpha}E^-((2m+1)c+\alpha_1,-z_2)E^-((2n-1)c\pm\alpha_2,z_2)$$
$$E^+((2m+1)c+\alpha_1,-z_2)E^+((2n-1)c\pm\alpha_2,z_2)\delta(-\frac{z_2}{z_1})$$
$$=z_2\epsilon^{(2m+2n)c\pm\alpha}z_2^{(2m+2n)c\pm\alpha}E^-((2m+2n)c\pm\alpha,\pm z_2)E^+((2m+2n)c\pm\alpha,\pm z_2)\delta(\frac{z_2}{z_1})$$
$$-z_2\epsilon^{(2m+2n)c\pm\alpha}z_2^{(2m+2n)c\pm\alpha}E^-((2m+2n)c\pm\alpha,\mp z_2)E^+((2m+2n)c\pm\alpha,\mp z_2)\delta(-\frac{z_2}{z_1})$$
$$=\pm x_\pm(\pm z_2,2m+2n)\delta(\frac{z_2}{z_1})\pm x_\pm(\mp z_2,2m+2n)\delta(-\frac{z_2}{z_1})$$
$$=\pm x_\pm(z_2,2m+2n)\delta(\pm\frac{z_2}{z_1})\pm x_\pm(-z_2,2m+2n)\delta(\mp\frac{z_2}{z_1}).$$

as required.

For (2.5.3). By Lemma 3.2.6 and Lemma 3.2.7, we have

$$x_+(z_1,2m)x_-(z_2,2n)$$

$$=z_1\epsilon^{2mc+\alpha}z_1^{2mc+\alpha}E^-(2mc+\alpha,z_1)E^+(2mc+\alpha,z_1)$$
$$\cdot z_2\epsilon^{2nc-\alpha}z_2^{2nc-\alpha}E^-(2nc-\alpha,z_2)E^+(2nc-\alpha,z_2)$$
$$=z_1z_2z_1^{\frac{1}{2}(2mc+\alpha,2nc-\alpha)}\epsilon^{(2m+2n)c}z_1^{2mc+\alpha}z_2^{2nc-\alpha}E^-(2mc+\alpha,z_1)E^-(2nc-\alpha,z_2)$$
$$\cdot E^+(2mc+\alpha,z_1)E^+(2nc-\alpha,z_2)(1+\frac{z_2}{z_1})^0(1-\frac{z_2}{z_1})^{-2}$$
$$=\epsilon^{(2m+2n)c}z_1^{2mc+\alpha}z_2^{2nc-\alpha}E^-(2mc+\alpha,z_1)E^-(2nc-\alpha,z_2)$$
$$\cdot E^+(2mc+\alpha,z_1)E^+(2nc-\alpha,z_2)(\frac{z_2}{z_1})(1-\frac{z_2}{z_1})^{-2}.$$

Similarly, we have

$$x_-(z_2, 2n)x_+(z_1, 2m)$$

$$= \epsilon^{(2m+2n)c} z_1^{2mc+\alpha} z_2^{2nc-\alpha} E^-(2mc+\alpha, z_1)E^-(2nc-\alpha, z_2)$$

$$\cdot E^+(2mc+\alpha, z_1)E^+(2nc-\alpha, z_2)(\frac{z_1}{z_2})(1-\frac{z_1}{z_2})^{-2}.$$

Therefore, we obtain

$$[x_+(z_1, 2m), x_-(z_2, 2n)]$$

$$= \epsilon^{(2m+2n)c} z_1^{2mc+\alpha} z_2^{2nc-\alpha} E^-(2mc+\alpha, z_1)E^-(2nc-\alpha, z_2)$$

$$\cdot E^+(2mc+\alpha, z_1)E^+(2nc-\alpha, z_2)\left((\frac{z_2}{z_1})(1-\frac{z_2}{z_1})^{-2} - (\frac{z_1}{z_2})(1-\frac{z_1}{z_2})^{-2}\right).$$

where $(\frac{z_2}{z_1})(1-\frac{z_2}{z_1})^{-2} - (\frac{z_1}{z_2})(1-\frac{z_1}{z_2})^{-2} = (D\delta)(\frac{z_2}{z_1})$ [see (3.2.9)]. Thus this and Lemma 1.5.1 give us

$$[x_+(z_1, 2m), x_-(z_2, 2n)]$$

$$= \epsilon^{(2m+2n)c} z_2^{(2m+2n)c} E^-((2m+2n)c, z_2)E^+((2m+2n)c, z_2)(D\delta)(\frac{z_2}{z_1})$$

$$+\epsilon^{(2m+2n)c} z_2^{(2m+2n)c} E^-((2m+2n)c, z_2)\{2mc(z_2)+\alpha(z_2)\}$$

$$\cdot E^+((2m+2n)c, z_2)\delta(\frac{z_2}{z_1})$$

$$= 2C_1(z_2, 2m+2n)(D\delta)(\frac{z_2}{z_1}) + 4mC_2(z_2, 2m+2n)\delta(\frac{z_2}{z_1})$$

$$+\alpha(z_2, 2m+2n)\delta(\frac{z_2}{z_1}).$$

as required.

For (2.5.4). As the above case, we have

$$x_+(z_1, 2m)x_-(z_2, 2n-1)$$

$$= z_1\epsilon^{2mc+\alpha} z_1^{2mc+\alpha} E^-(2mc+\alpha, z_1)E^+(2mc+\alpha, z_1) z_2\epsilon^{(2n-1)c-\alpha} z_2^{(2n-1)c-\alpha}$$

$$\cdot E^-((2n-1)c-\alpha_2, z_2)W(z_2)E^+((2n-1)c-\alpha_2, z_2)$$

$$= z_1 z_2 \epsilon(2mc+\alpha, (2n-1)c-\alpha) z_1^{\frac{1}{2}(2mc+\alpha,(2n-1)c-\alpha)} \epsilon^{(2m+2n-1)c} z_1^{2mc+\alpha} z_2^{(2n-1)c-\alpha}$$

$$\cdot E^-(2mc+\alpha, z_1)E^-((2n-1)c-\alpha_2, z_2)W(z_2)E^+(2mc+\alpha, z_1)E^+((2n-1)c-\alpha_2, z_2)$$

$$\cdot(1+\frac{z_2}{z_1})^{-1}(1-\frac{z_2}{z_1})^{-1}$$

$$= -\epsilon^{(2m+2n-1)c} z_1^{2mc+\alpha} z_2^{(2n-1)c-\alpha} E^-(2mc+\alpha, z_1) E^-((2n-1)c-\alpha_2, z_2) W(z_2)$$

$$\cdot E^+(2mc+\alpha, z_1) E^+((2n-1)c-\alpha_2, z_2) \frac{z_2}{z_1}(1-\frac{z_2^2}{z_1^2})^{-1}.$$

Similarly, we have

$$x_-(z_2, 2n-1) x_+(z_1, 2m)$$

$$= \epsilon^{(2m+2n-1)c} z_1^{2mc+\alpha} z_2^{(2n-1)c-\alpha} E^-(2mc+\alpha, z_1) E^-((2n-1)c-\alpha_2, z_2) W(z_2)$$

$$\cdot E^+(2mc+\alpha, z_1) E^+((2n-1)c-\alpha_2, z_2) \frac{z_1}{z_2}(1-\frac{z_1^2}{z_2^2})^{-1}.$$

Therefore, we obtain

$$[x_+(z_1, 2m), x_-(z_2, 2n-1)]$$

$$= -\epsilon^{(2m+2n-1)c} z_1^{2mc+\alpha} z_2^{(2n-1)c-\alpha} E^-(2mc+\alpha, z_1) E^-((2n-1)c-\alpha_2, z_2) W(z_2)$$

$$\cdot E^+(2mc+\alpha, z_1) E^+((2n-1)c-\alpha_2, z_2) S(z_1, z_2).$$

where [cf. (3.2.9)]

$$S(z_1, z_2) = \frac{z_2}{z_1}(1-\frac{z_2^2}{z_1^2})^{-1} + \frac{z_1}{z_2}(1-\frac{z_1^2}{z_2^2})^{-1} = \sum_{k \in 2\mathbf{Z}+1} (\frac{z_2}{z_1})^k$$

$$= \frac{1}{2}(\delta(\frac{z_2}{z_1}) - \delta(-\frac{z_2}{z_1})).$$

Hence, by Lemma 1.5.1, we obtain

$$[x_+(z_1, 2m), x_-(z_2, 2n-1)]$$

$$= -\frac{1}{2}\epsilon^{(2m+2n-1)c} z_2^{(2m+2n-1)c} E^-((2m+2n-1)c+\alpha_1, z_2) W(z_2)$$

$$\cdot E^+((2m+2n-1)c+\alpha_1, z_2) \delta(\frac{z_2}{z_1})$$

$$-\frac{1}{2}\epsilon^{(2m+2n-1)c} z_2^{(2m+2n-1)c} E^-((2m+2n-1)c+\alpha_1, -z_2) W(-z_2)$$

$$\cdot E^+((2m+2n-1)c+\alpha_1, -z_2) \delta(-\frac{z_2}{z_1})$$

$$= \frac{1}{2}\alpha(z_2, 2m+2n-1)\delta(\frac{z_2}{z_1}) + \frac{1}{2}\alpha(-z_2, 2m+2n-1)\delta(-\frac{z_2}{z_1}).$$

where we have used the fact: $W(-z_2) = -W(z_2)$.

For (2.5.5). We have

$$x_+(z_1, 2m+1)x_-(z_2, 2n)$$

$$= z_1\epsilon^{(2m+1)c+\alpha} z_1^{(2m+1)c+\alpha} E^-((2m+1)c+\alpha_2, z_1)W(z_1)E^+((2m+1)c+\alpha_2, z_1)$$

$$\cdot z_2\epsilon^{2nc-\alpha} z_2^{2nc-\alpha} E^-(2nc-\alpha, z_2)E^+(2nc-\alpha, z_2)$$

$$= z_1z_2\epsilon((2m+1)c+\alpha, 2nc-\alpha)z_1^{\frac{1}{2}((2m+1)c+\alpha, 2nc-\alpha)}\epsilon^{(2m+2n+1)c} z_1^{(2m+1)c+\alpha} z_2^{2nc-\alpha}$$

$$\cdot E^-((2m+1)c+\alpha_2, z_1)E^-(2nc-\alpha, z_2)W(z_1)E^+((2m+1)c+\alpha_2, z_1)E^+(2nc-\alpha, z_2)$$

$$\cdot(1+\frac{z_2}{z_1})^{-1}(1-\frac{z_2}{z_1})^{-1}$$

$$= \epsilon^{(2m+2n+1)c} z_1^{(2m+1)c+\alpha} z_2^{2nc-\alpha} E^-((2m+1)c+\alpha_2, z_1)E^-(2nc-\alpha, z_2)W(z_1)$$

$$\cdot E^+((2m+1)c+\alpha_2, z_1)E^+(2nc-\alpha, z_2)\frac{z_2}{z_1}(1-\frac{z_2^2}{z_1^2})^{-1}.$$

Similarly, we have

$$x_-(z_2, 2n)x_+(z_1, 2m+1)$$

$$= -\epsilon^{(2m+2n+1)c} z_1^{(2m+1)c+\alpha} z_2^{2nc-\alpha} E^-((2m+1)c+\alpha_2, z_1)E^-(2nc-\alpha, z_2)W(z_1)$$

$$\cdot E^+((2m+1)c+\alpha_2, z_1)E^+(2nc-\alpha, z_2)\frac{z_1}{z_2}(1-\frac{z_1^2}{z_2^2})^{-1}.$$

Therefore, we have

$$[x_+(z_1, 2m+1), x_-(z_2, 2n)]$$

$$= \epsilon^{(2m+2n+1)c} z_1^{(2m+1)c+\alpha} z_2^{2nc-\alpha} E^-((2m+1)c+\alpha_2, z_1)E^-(2nc-\alpha, z_2)W(z_1)$$

$$\cdot E^+((2m+1)c+\alpha_2, z_1)E^+(2nc-\alpha, z_2)\frac{1}{2}(\delta(\frac{z_2}{z_1})-\delta(-\frac{z_2}{z_1}))$$

$$= \frac{1}{2}\epsilon^{(2m+2n+1)c} z_2^{(2m+2n+1)c} E^-((2m+2n+1)c-\alpha_1, z_2)W(z_2)E^+((2m+2n+1)c-\alpha_1, z_2)\delta(\frac{z_2}{z_1})$$

$$-\frac{1}{2}\epsilon^{(2m+2n+1)c} z_2^{(2m+2n+1)c} E^-((2m+2n+1)c-\alpha_1, z_2)W(-z_2)$$

$$\cdot E^+((2m+2n+1)c-\alpha_1, z_2)\delta(-\frac{z_2}{z_1})$$

$$= \frac{1}{2}\alpha(-z_2, 2m+2n+1)(\delta(\frac{z_2}{z_1})+\delta(-\frac{z_2}{z_1})).$$

as required.

For (2.5.6). We have

$$x_+(z_1, 2m+1)x_-(z_2, 2n-1)$$

$$= z_1\epsilon^{(2m+1)c+\alpha} z_1^{(2m+1)c+\alpha} E^-((2m+1)c+\alpha_2, z_1)W(z_1)E^+((2m+1)c+\alpha_2, z_1)$$

$$\cdot z_2\epsilon^{(2n-1)c-\alpha} z_2^{(2n-1)c-\alpha} E^-((2n-1)c-\alpha_2, z_2)W(z_2)E^+((2n-1)c-\alpha_2, z_2)$$

$$= z_1z_2\epsilon((2m+1)c+\alpha, (2n-1)c-\alpha)z_1^{\frac{1}{2}((2m+1)c+\alpha, (2n-1)c-\alpha)}\epsilon^{(2m+2n)c} z_1^{(2m+1)c+\alpha} z_2^{(2n-1)c-\alpha}$$

$$\cdot E^-((2m+1)c+\alpha_2, z_1)E^-((2n-1)c-\alpha_2, z_2)(1+\frac{z_2}{z_1})^{-1}(1-\frac{z_2}{z_1})^{-1}W(z_1)W(z_2)$$

$$\cdot E^+((2m+1)c+\alpha_2, z_1)E^+((2n-1)c-\alpha_2, z_2)$$

$$= -\epsilon^{(2m+2n)c} z_1^{(2m+1)c+\alpha} z_2^{(2n-1)c-\alpha} E^-((2m+1)c+\alpha_2, z_1)E^-((2n-1)c-\alpha_2, z_2)$$

$$\cdot\frac{z_2}{z_1}(1-\frac{z_2^2}{z_1^2})^{-1}W(z_1)W(z_2)E^+((2m+1)c+\alpha_2, z_1)E^+((2n-1)c-\alpha_2, z_2).$$

Similarly, we have

$$x_-(z_2, 2n-1)x_+(z_1, 2m+1)$$

$$= -\epsilon^{(2m+2n)c} z_1^{(2m+1)c+\alpha} z_2^{(2n-1)c-\alpha} E^-((2m+1)c+\alpha_2, z_1)E^-((2n-1)c-\alpha_2, z_2)$$

$$\cdot\frac{z_1}{z_2}(1-\frac{z_1^2}{z_2^2})^{-1}W(z_2)W(z_1)E^+((2m+1)c+\alpha_2, z_1)E^+((2n-1)c-\alpha_2, z_2).$$

Thus, by Lemma 3.2.4 and Lemma 1.5.1, we obtain

$$[x_+(z_1, 2m+1), x_-(z_2, 2n-1)]$$

$$= \frac{1}{2}\epsilon^{(2m+2n)c} z_1^{(2m+1)c+\alpha} z_2^{(2n-1)c-\alpha} E^-((2m+1)c+\alpha_2, z_1)E^-((2n-1)c-\alpha_2, z_2)$$

$$\cdot E^+((2m+1)c+\alpha_2, z_1)E^+((2n-1)c-\alpha_2, z_2)((D\delta)(\frac{z_2}{z_1})+(D\delta)(-\frac{z_2}{z_1}))$$

$$= \frac{1}{2}\epsilon^{(2m+2n)c} z_2^{(2m+2n)c} E^-((2m+2n)c, z_2)E^+((2m+2n)c, z_2)((D\delta)(\frac{z_2}{z_1})+(D\delta)(-\frac{z_2}{z_1}))$$

$$+\frac{1}{2}\epsilon^{(2m+2n)c} z_2^{(2m+2n)c} E^-((2m+2n)c, z_2)\{(2m+1)c(z_2)+\alpha_2(z_2)\}E^+((2m+2n)c, z_2)$$

$$\cdot(\delta(\frac{z_2}{z_1})+\delta(-\frac{z_2}{z_1}))$$

$$= C_1(z_2, 2m+2n)((D\delta)(\frac{z_2}{z_1}) + (D\delta)(-\frac{z_2}{z_1}))$$
$$+(2m+1)C_2(z_2, 2m+1)(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1}))$$
$$+\frac{1}{4}(\alpha(z_2, 2m+2n) + \alpha(-z_2, 2m+2n))(\delta(\frac{z_2}{z_1}) + \delta(-\frac{z_2}{z_1})).$$

as required.

For (2.5.14). We have

$$D_z C_1(z, 2n) = D_z\{\frac{1}{2}\epsilon^{2nc} z^{2nc} E^-(2nc, z) E^+(2nc, z)\}$$
$$= \frac{1}{2}\epsilon^{2nc} z^{2nc} E^-(2nc, z) 2nc(z) E^+(2nc, z) = 2nC_2(z, 2n).$$

as required.

□

To finish the proof of Theorem 3.2.8, we remain to check (2.5.2). For this purpose, we divide the argument into three cases

(i). $[x_\pm(z_1, 2m), x_\pm(z_2, 2n)] = 0.$

(ii). $[x_\pm(z_1, 2m), x_\pm(z_2, 2n-1)] = 0.$

(iii). $[x_\pm(z_1, 2m+1), x_\pm(z_2, 2n-1)] = 0.$

Proof of (i). Since

$$x_\pm(z_1, 2m) x_\pm(z_2, 2n)$$
$$= z_1 \epsilon^{2mc\pm\alpha} z_1^{2mc\pm\alpha} E^-(2mc\pm\alpha, z_1) E^+(2mc\pm\alpha, z_1)$$
$$\cdot z_2 \epsilon^{2nc\pm\alpha} z_2^{2nc\pm\alpha} E^-(2nc\pm\alpha, z_2) E^+(2nc\pm\alpha, z_2)$$
$$= z_1 z_2 z_1^2 \epsilon^{(2m+2n)c\pm 2\alpha} z_1^{2mc\pm\alpha} z_2^{2nc\pm\alpha} E^-(2mc\pm\alpha, z_1) E^-(2nc\pm\alpha, z_2)$$
$$\cdot E^+(2mc\pm\alpha, z_1) E^+(2nc\pm\alpha, z_2)(1-\frac{z_2}{z_1})^2.$$

we obtain from this

$$[x_\pm(z_1, 2m), x_\pm(z_2, 2n)]$$
$$= z_1 z_2 \epsilon^{(2m+2n)c\pm 2\alpha} z_1^{2mc\pm\alpha} z_2^{2nc\pm\alpha} E^-(2mc\pm\alpha, z_1) E^-(2nc\pm\alpha, z_2)$$
$$\cdot E^+(2mc\pm\alpha, z_1) E^+(2nc\pm\alpha, z_2)\{z_1^2(1-\frac{z_2}{z_1})^2 - z_2^2(1-\frac{z_1}{z_2})^2\} = 0.$$

as required.

Proof of (ii). we have

$$x_{\pm}(z_1, 2m)x_{\pm}(z_2, 2n-1)$$

$$= z_1\epsilon^{2mc\pm\alpha}z_1^{2mc\pm\alpha}E^-(2mc\pm\alpha, z_1)E^+(2mc\pm\alpha, z_1)$$

$$\cdot z_2\epsilon^{(2n-1)c\pm\alpha}z_2^{(2n-1)c\pm\alpha}E^-((2n-1)c\pm\alpha_2, z_2)W(z_2)E^+((2n-1)c\pm\alpha_2, z_2)$$

$$= z_1z_2\epsilon(2mc\pm\alpha, (2n-1)c\pm\alpha)z_1^2\epsilon^{(2m+2n-1)c\pm 2\alpha}z_1^{2mc\pm\alpha}z_2^{(2n-1)c\pm\alpha}E^-(2mc\pm\alpha, z_1)$$

$$E^-((2n-1)c\pm\alpha_2, z_2)W(z_2)E^+(2mc\pm\alpha, z_1)E^+((2n-1)c\pm\alpha_2, z_2)(1+\frac{z_2}{z_1})(1-\frac{z_2}{z_1}).$$

Similarly, we have

$$x_{\pm}(z_2, 2n-1)x_{\pm}(z_1, 2m)$$

$$= z_1z_2\epsilon((2n-1)c\pm\alpha, 2mc\pm\alpha)z_2^2\epsilon^{(2m+2n-1)c\pm 2\alpha}z_1^{2mc\pm\alpha}z_2^{(2n-1)c\pm\alpha}E^-(2mc\pm\alpha, z_1)$$

$$E^-((2n-1)c\pm\alpha_2, z_2)W(z_2)E^+(2mc\pm\alpha, z_1)E^+((2n-1)c\pm\alpha_2, z_2)(1+\frac{z_1}{z_2})(1-\frac{z_1}{z_2}).$$

Therefore, we obtain

$$[x_{\pm}(z_1, 2m), x_{\pm}(z_2, 2n-1)]$$

$$= z_1z_2\epsilon^{(2m+2n-1)c\pm 2\alpha}z_1^{2mc\pm\alpha}z_2^{(2n-1)c\pm\alpha}E^-(2mc\pm\alpha, z_1)E^-((2n-1)c\pm\alpha_2, z_2)$$

$$\cdot W(z_2)E^+(2mc\pm\alpha, z_1)E^+((2n-1)c\pm\alpha_2, z_2)T(z_1, z_2),$$

where

$$T(z_1, z_2) = \epsilon(2mc\pm\alpha, (2n-1)c\pm\alpha)z_1^2(1-\frac{z_2^2}{z_1^2})$$

$$-\epsilon((2n-1)c\pm\alpha, 2mc\pm\alpha)z_2^2(1-\frac{z_1^2}{z_2^2})$$

$$= -(z_1^2-z_2^2)-(z_2^2-z_1^2) = 0,$$

which completes the proof of (ii).

Proof of (iii). we have

$$x_{\pm}(z_1, 2m+1)x_{\pm}(z_2, 2n-1)$$

$$= z_1\epsilon^{(2m+1)c\pm\alpha}z_1^{(2m+1)c\pm\alpha}E^-((2m+1)c\pm\alpha_2, z_1)W(z_1)E^+((2m-1)c\pm\alpha_2, z_1)$$

$$\cdot z_2 \epsilon^{(2n-1)c\pm\alpha} z_2^{(2n-1)c\pm\alpha} E^-((2n-1)c \pm \alpha_2, z_2) W(z_2) E^+((2n-1)c \pm \alpha_2, z_2)$$

$$= z_1 z_2 z_1^2 (-1) \epsilon^{(2m+2n)c\pm 2\alpha} z_1^{(2m+1)c\pm\alpha} z_2^{(2n-1)c\pm\alpha} E^-((2m+1)c \pm \alpha_2, z_1)$$

$$\cdot E^-((2n-1)c \pm \alpha_2, z_2)(1 + \frac{z_2}{z_1})(1 - \frac{z_2}{z_1}) W(z_1) W(z_2)$$

$$\cdot E^+((2m+1)c \pm \alpha_2, z_1) E^+((2n-1)c \pm \alpha_2, z_2).$$

Therefore, this gives us

$$[x_\pm(z_1, 2m+1), x_\pm(z_2, 2n-1)]$$

$$= -z_1 z_2 \epsilon^{(2m+2n)c\pm 2\alpha} z_1^{(2m+1)c\pm\alpha} z_2^{(2n-1)c\pm\alpha} E^-((2m+1)c \pm \alpha_2, z_1) E^-((2n-1)c \pm \alpha_2, z_2)$$

$$U(z_1, z_2) E^+((2m+1)c \pm \alpha_2, z_1) E^+((2n-1)c \pm \alpha_2, z_2),$$

where

$$\begin{aligned} U(z_1, z_2) &= z_1^2 (1 - \frac{z_2^2}{z_1^2}) W(z_1) W(z_2) - z_2^2 (1 - \frac{z_1^2}{z_2^2}) W(z_2) W(z_1) \\ &= (z_1^2 - z_2^2)(W(z_1)W(z_2) + W(z_2)W(z_1)) \\ &= (z_1^2 - z_2^2)(\delta(-\frac{z_2}{z_1}) - \delta(\frac{z_2}{z_1})) = 0, \end{aligned}$$

as required, where we have used Lemma 3.2.3.

Therefore, we completed the proof of Theorem 3.2.8.

§3.4 Extended TKK Algebra

Recall the $\mathbf{Z}^2$–grading of the TKK algebra $\mathcal{G}(\mathcal{T})$, given by (2.4.1). We extend this gradation to the universal central extension $\hat{\mathcal{G}}(\mathcal{T})$ of the TKK algebra $\mathcal{G}(\mathcal{T})$ by letting

$$\hat{\mathcal{G}}(\mathcal{T}) = \dot{+}_{\sigma \in \mathbf{Z}^2} \hat{\mathcal{G}}^\sigma(\mathcal{T}), \tag{3.4.1}$$

where $\hat{\mathcal{G}}^\sigma(\mathcal{T}) = \hat{\mathcal{G}}^\sigma_+(\mathcal{T}) \dot{+} \hat{\mathcal{G}}^\sigma_0(\mathcal{T}) \dot{+} \hat{\mathcal{G}}^\sigma_-(\mathcal{T})$, and

$$\hat{\mathcal{G}}^\sigma_\pm(\mathcal{T}) = \mathcal{G}^\sigma_\pm(\mathcal{T}),$$

$$\hat{\mathcal{G}}_0^\sigma(\mathcal{T}) = \begin{cases} \mathbb{C}\alpha \otimes x^\sigma \dotplus \mathbb{C}c_1(\sigma) \dotplus \mathbb{C}c_2(\sigma), & \text{if } \sigma \in S_0, \\ \mathbb{C}\alpha \otimes x^\sigma, & \text{if } \sigma \in S_1 \cup S_2, \\ \mathbb{C}[L_{x^{\delta_1}}, L_{x^{\sigma-\delta_1}}], & \text{if } \sigma \in S^\perp. \end{cases}$$

Therefore, we may extend the degree derivations d_i $(i = 1, 2)$ of the $\mathcal{G}(\mathcal{T})$ to $\hat{\mathcal{G}}(\mathcal{T})$. That is

$$d_i y = (\sigma \cdot \delta_i) y, \tag{3.4.2}$$

for $i = 1, 2$, $y \in \hat{\mathcal{G}}^\sigma(\mathcal{T})$, and $\sigma \in \mathbf{Z}^2$.

Now we define the extended TKK algebra $\check{\mathcal{G}}(\mathcal{T})$ by taking the semi-direct product of $\hat{\mathcal{G}}(\mathcal{T})$ with $\mathcal{D} := \mathbb{C}d_1 + \mathbb{C}d_2$. That is

$$\check{\mathcal{G}}(\mathcal{T}) := \hat{\mathcal{G}}(\mathcal{T}) \dotplus \mathcal{D},$$

with the Lie product given by (2.2.2) and

$$[\mathcal{D}, \mathcal{D}] = \{0\}, \tag{3.4.3}$$

$$[d_i, y] = d_i y,$$

for $i = 1, 2$, and $y \in \hat{\mathcal{G}}(\mathcal{T})$.

Let $K := \{y \mid y \in HC_1(\mathcal{T}) \cap \hat{\mathcal{G}}^\sigma(\mathcal{T}), \text{ for some } \sigma \in \mathbf{Z}^2 \setminus \{0\}\}$. Then it is clear that K is an ideal of $\check{\mathcal{G}}(\mathcal{T})$, and

$$\check{\mathcal{G}}(\mathcal{T})/K \simeq \check{\mathcal{L}}(\mathcal{T}),$$

where $\check{\mathcal{L}}(\mathcal{T})$ is the extended affine Lie algebra of type A_1, [cf. III.2 in [AABGP]].

In this section, we want to extend the action of $\hat{\mathcal{G}}(\mathcal{T})$ on V to $\check{\mathcal{G}}(\mathcal{T})$. For this purpose, we introduce the Segal operator L_n by

$$L_n = \frac{1}{4} \sum_{j \in \mathbf{Z}} : \alpha(n-j)\alpha(j) : + \frac{1}{2} \sum_{j \in 2\mathbf{Z}} : d(n-j)c(j) : \tag{3.4.4}$$

for $n \in \mathbf{Z}$, where : : is the usual normal ordering defined by

$$: a(j)b(k) := \begin{cases} a(j)b(k), & \text{if } j \le k, \\ b(k)a(j), & \text{if } j > k. \end{cases}$$

The following two identities (3.4.5) and (3.4.6) can be checked by a routine argument. [cf. [GO], [KR] or [KMPS]]

$$[a(m),\ L_n] = ma(m+n), \tag{3.4.5}$$

where $a \in \mathbf{C}\alpha$, $m \in \mathbf{Z}$, or $a \in \mathbf{C}c + \mathbf{C}d$, $m \in 2\mathbf{Z}$.

$$[L_m, L_n] = (m-n)L_{m+n} + P(m)\delta_{m+n,0}, \tag{3.4.6}$$

for $m, n \in \mathbf{Z}$, where

$$P(m) = \begin{cases} \frac{m^3-m}{12}, & \text{if } m \in 2\mathbf{Z}+1, \\ \frac{2m^3-5m}{12}, & \text{if } m \in 2\mathbf{Z}. \end{cases}$$

Let $\tilde{d}_1 = L_0 + T_0$. [see (3.2.5) for the definition of T_n]. We have

Lemma 3.4.1 We have

$$\tilde{d}_1(e^\gamma \otimes a_k(-n_k)\cdots a_1(-n_1) \otimes \omega_{-m_l} \wedge \cdots \wedge \omega_{-m_1})$$
$$= \left(\frac{1}{4}(\gamma,\gamma) + \sum_{i=1}^{k} n_i + \sum_{j=1}^{l} m_j\right) e^\gamma \otimes a_k(-n_k)\cdots a_1(-n_1) \otimes \omega_{-m_l} \wedge \cdots \wedge \omega_{-m_1},$$

where $a_i \in \mathbf{C}\alpha + \mathbf{C}c + \mathbf{C}d$, $n_i \in \mathbf{Z}_+$, $m_j \in 2\mathbf{N}+1$, $\gamma \in \Gamma$.

Proof. Since

$$\tilde{d}_1 = L_0 + T_0$$
$$= \frac{1}{4}\sum_{j\in\mathbf{Z}} : \alpha(-j)\alpha(j) : + \frac{1}{2}\sum_{j\in 2\mathbf{Z}} : d(-j)c(j) : - \frac{1}{2}\sum_{j=1}^{\infty} j\omega_{-j}\omega_j,$$

we have

$$\tilde{d}_1.e^\gamma \otimes 1 \otimes 1 \tag{3.4.7}$$
$$= (\frac{1}{4}\alpha(0)\alpha(0) + \frac{1}{2}d(0)c(0))e^\gamma \otimes 1 \otimes 1$$
$$= \left(\frac{1}{4}(\frac{(\alpha,\gamma)}{2})^2 + \frac{1}{2}\frac{(d,\gamma)}{2}\frac{(c,\gamma)}{2}\right) e^\gamma \otimes 1 \otimes 1$$
$$= \frac{(\gamma,\gamma)}{4}e^\gamma \otimes 1 \otimes 1.$$

Set

$$\Omega_{i,j} := a_i(-n_i)\cdots a_1(-n_1) \otimes \omega_{-m_j} \wedge \cdots \wedge \omega_{-m_1}.$$

for $i, j \geq 1$, and

$$\Omega_{0,j} := 1 \otimes \omega_{-m_j} \wedge \cdots \wedge \omega_{-m_1},$$

$$\Omega_{i,0} := a_i(-n_i) \cdots a_1(-n_1) \otimes 1,$$

and $\Omega_{0,0} := 1 \otimes 1$.

Now we prove this Lemma by induction on $k+l$, for $k, l \in \mathbf{N}$. The case for $k+l=0$ is given by (3.4.7). Thus we may assume that $k+l \geq 1$.

If $k \geq 1$, we have

$$\tilde{d}_1.(\epsilon^\gamma \otimes a_k(-n_k) \cdots a_1(-n_1) \otimes \omega_{-m_l} \wedge \cdots \wedge \omega_{-m_1})$$

$$= \tilde{d}_1.(a_k(-n_k).\epsilon^\gamma \otimes \Omega_{k-1,l})$$

$$= [\tilde{d}_1, a_k(-n_k)].(\epsilon^\gamma \otimes \Omega_{k-1,l}) + a_k(-n_k).\tilde{d}_1.(\epsilon^\gamma \otimes \Omega_{k-1,l})$$

$$= n_k a_k(-n_k).(\epsilon^\gamma \otimes \Omega_{k-1,l})$$

$$+(\frac{1}{4}(\gamma,\gamma) + \sum_{i=1}^{k-1} n_i + \sum_{j=1}^{l} m_j) a_k(-n_k).(\epsilon^\gamma \otimes \Omega_{k-1,l})$$

$$= (\frac{1}{4}(\gamma,\gamma) + \sum_{i=1}^{k} n_i + \sum_{j=1}^{l} m_j)\epsilon^\gamma \otimes \Omega_{k,l},$$

as required.

Similarly, if $l \geq 1$, we have

$$\tilde{d}_1.(\epsilon^\gamma \otimes a_k(-n_k) \cdots a_1(-n_1) \otimes \omega_{-m_l} \wedge \cdots \wedge \omega_{-m_1})$$

$$= \tilde{d}_1.(\omega_{-m_l}.\epsilon^\gamma \otimes \Omega_{k,l-1})$$

$$= [\tilde{d}_1, \omega_{-m_l}].(\epsilon^\gamma \otimes \Omega_{k,l-1}) + \omega_{-m_l}.\tilde{d}_1.(\epsilon^\gamma \otimes \Omega_{k,l-1})$$

$$= m_l \omega_{-m_l}.(\epsilon^\gamma \otimes \Omega_{k-1,l})$$

$$+(\frac{1}{4}(\gamma,\gamma) + \sum_{i=1}^{k} n_i + \sum_{j=1}^{l-1} m_j)\omega_{-m_l}.(\epsilon^\gamma \otimes \Omega_{k,l-1})$$

$$= (\frac{1}{4}(\gamma,\gamma) + \sum_{i=1}^{k} n_i + \sum_{j=1}^{l} m_j)\epsilon^\gamma \otimes \Omega_{k,l}.$$

as required. This completes the proof of the Lemma.

$\square$

We apply Lemma 3.4.1 to define a $\mathbf{Z}$-gradation on the Fock space V:

$$V = \oplus_{n \in \mathbf{Z}} V^n,$$

where $V^n = \{v \in V | \ \tilde{d}_1.v = nv\}$.

Notice that, with this gradation, the operators $\alpha(m), c(m), d(m)$ and ω_m act on V with degree $-m$, for $m \in \mathbf{Z}$. Moreover, we have

Lemma 3.4.2 Let $\nu \in \Gamma$. Let z be any formal variable. Then

$$[\tilde{d}_1, \ e^\nu] = e^\nu(\frac{(\nu,\nu)}{4} + \nu(0)).$$

$$[\tilde{d}_1, \ z^\nu] = 0.$$

Proof. Let $e^\mu \otimes a$ be an element of V which is homogeneous of degree $\frac{(\mu,\mu)}{4} + m$, for some $m \in \mathbf{Z}$. We have

$$[\tilde{d}_1, \ e^\nu].e^\mu \otimes a = \tilde{d}_1 \epsilon(\nu,\mu) e^{\nu+\mu} \otimes a - e^\nu \left(\frac{(\mu,\mu)}{4} + m\right) e^\mu \otimes a$$

$$= \epsilon(\nu,\mu)\left(\frac{(\nu+\mu,\nu+\mu)}{4} + m\right) e^{\nu+\mu} \otimes a - \left(\frac{(\mu,\mu)}{4} + m\right) e^\nu.e^\mu \otimes a$$

$$= \left(\frac{(\nu,\nu)}{4} + \nu(0)\right) e^\nu.e^\mu \otimes a.$$

This proves the first identity. The second identity is clear since both $z^{\nu(0)}$ and $\tilde{d}_1$ act as scalar multiplications on V.

$\square$

Proposition 3.4.3 Let $X(z,m)$ be any vertex operator given by (3.2.12). Then we have

$$[\tilde{d}_1, \ X(z,m)] = D_z X(z,m),$$

for $m \in \mathbf{Z}$.

Proof. It is clear that the result of this Proposition follows from the following four identities.

$$[\tilde{d}_1, \ \beta(z)] = D_z\beta(z), \tag{3.4.8}$$

$$[\tilde{d}_1, \ W(z)] = D_zW(z), \tag{3.4.9}$$

$$[\tilde{d}_1, \ z^{\frac{(\beta,\beta)}{4}}e^{\beta}z^{\beta}] = D_z(z^{\frac{(\beta,\beta)}{4}}e^{\beta}z^{\beta}), \tag{3.4.10}$$

$$[\tilde{d}_1, \ E^{\pm}(\gamma, z)] = D_zE^{\pm}(\gamma, z), \tag{3.4.11}$$

where $\beta \in \mathbf{Z}\alpha + \mathbf{Z}c$, $\gamma \in \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2 + \mathbf{Z}c$.

Moreover, it is easy to check the identities (3.4.8), (3.4.9) and (3.4.11) by using (3.4.5), Proposition 3.2.2 and (3.2.10). To prove (3.4.10), we apply Lemma 3.4.2, and obtain

$$[\tilde{d}_1, \ z^{\frac{(\beta,\beta)}{4}}e^{\beta}z^{\beta}] = z^{\frac{(\beta,\beta)}{4}}[\tilde{d}_1, \ e^{\beta}]z^{\beta}$$
$$= \left(\frac{(\beta,\beta)}{4} + \beta(0)\right)z^{\frac{(\beta,\beta)}{4}}e^{\beta}z^{\beta} = D_z\left(z^{\frac{(\beta,\beta)}{4}}e^{\beta}z^{\beta}\right),$$

as required.

□

The previous Proposition also give us

Theorem 3.4.4 The moment operators of the vertex operators given by (3.2.12), and the operators $\tilde{d}_1$, $\tilde{d}_2 := \frac{1}{2}d(0)$ span a Lie algebra which is isomorphic to the extended TKK algebra $\tilde{\mathcal{G}}(\mathcal{T}) = \hat{\mathcal{G}}(\mathcal{T}) \dotplus \mathcal{D}$.

§3.5 Realizations of the Toroidal Lie Algebra of Type A_1

Recall from Chapter 1 that the (homogenerous) toroidal Lie algebra $\hat{\mathcal{G}}$ of type A_1 with two variables is spanned by the following elements

$$x_+ \odot s^m t^n, \quad x_- \odot s^m t^n, \quad \alpha \odot s^m t^n,$$

$$\overline{s^m t^n (s^{-1}ds)}, \quad \overline{s^m t^n (t^{-1}dt)},$$

for $m, n \in \mathbf{Z}$, with the following commutation relations:

(R1).

$$[x_+ \otimes s^{m_1}t^{n_1}, \ x_- \otimes s^{m_2}t^{n_2}] = \alpha \otimes s^{m_1+m_2}t^{n_1+n_2} + m_1 \overline{s^{m_1+m_2}t^{n_1+n_2}(s^{-1}ds)}$$

$$+n_1 \overline{s^{m_1+m_2}t^{n_1+n_2}(t^{-1}dt)}.$$

(R2).

$$[\alpha \otimes s^{m_1}t^{n_1}, \ x_+ \otimes s^{m_2}t^{n_2}] = 2x_+ \otimes s^{m_1+m_2}t^{n_1+n_2}.$$

(R3).

$$[\alpha \otimes s^{m_1}t^{n_1}, \ x_- \otimes s^{m_2}t^{n_2}] = -2x_- \otimes s^{m_1+m_2}t^{n_1+n_2}.$$

(R4).

$$[\alpha \otimes s^{m_1}t^{n_1}, \ \alpha \otimes s^{m_2}t^{n_2}] = 2m_1 \overline{s^{m_1+m_2}t^{n_1+n_2}(s^{-1}ds)} + 2n_1 \overline{s^{m_1+m_2}t^{n_1+n_2}(t^{-1}dt)}.$$

(R5).

$$\overline{s^m t^n (s^{-1}ds)}, \ \overline{s^m t^n (t^{-1}dt)} \text{ are central, and satisfy}$$

$$m\overline{s^m t^n (s^{-1}ds)} + n\overline{s^m t^n (t^{-1}dt)} = 0.$$

where $m, n, m_1, m_2, n_1, n_2 \in \mathbf{Z}$.

Moreover, Proposition 1.2.1 tells us that the toroidal Lie algebra $\hat{\mathcal{G}}$ contains a proper subalgebra $\hat{\mathcal{G}}_1$ which is isomorphic to $\hat{\mathcal{G}}$. Indeed this subalgebra $\hat{\mathcal{G}}_1$ gives the so called principal toroidal Lie algebra of type A_1 which we have studied in Chapter 1. That is, $\hat{\mathcal{G}}_1$ is spanned by the elements of the form

$$x_+ \otimes s^m t^{2n+1}, \quad x_- \otimes s^m t^{2n+1}, \quad \alpha \otimes s^m t^{2n},$$

$$\overline{s^m t^{2n} (s^{-1}ds)}, \quad \overline{s^m t^{2n} (t^{-1}dt)},$$

for $m, n \in \mathbf{Z}$, with the commutation relations given by $(R_1) - (R_5)$.

As consequences to Theorem 3.2.8, we have the following two corollaries:

Corollary 3.5.1 The operators

$$x_{\pm}(2m,2n),\quad \alpha(2m,2n),\quad C_i(2m,2n),$$

for $i=1,2, m,n\in\mathbf{Z}$, acting on V form a Lie algebra which is isomorphic to the homogenerous toroidal Lie algebra $\hat{\mathcal{G}}$ of type A_1 via the homomorphism ϕ_1:

$$x_+(2m,2n)\mapsto x_+\otimes s^m t^n,$$

$$x_-(2m,2n)\mapsto x_-\otimes s^m t^n,$$

$$\alpha(2m,2n)\mapsto \alpha\otimes s^m t^n,$$

$$C_1(2m,2n)\mapsto \frac{1}{4}\overline{s^m t^n(s^{-1}ds)},$$

$$C_2(2m,2n)\mapsto \frac{1}{4}\overline{s^m t^n(t^{-1}dt)}.$$

□

Corollary 3.5.2 The operators

$$x_{\pm}(2m,2n+1),\quad \alpha(2m,2n),\quad C_i(2m,2n),$$

for $i=1,2, m,n\in\mathbf{Z}$, acting on V form a Lie algebra which is isomorphic to the principal toroidal Lie algebra $\hat{\mathcal{G}}_1$ of type A_1 via the homomorphism ϕ_2:

$$x_+(2m,2n+1)\mapsto x_+\otimes s^m t^{2n+1},$$

$$x_-(2m,2n+1)\mapsto x_-\otimes s^m t^{2n+1},$$

$$\alpha(2m,2n)\mapsto \alpha\otimes s^m t^{2n},$$

$$C_1(2m,2n)\mapsto \frac{1}{4}\overline{s^m t^{2n}(s^{-1}ds)},$$

$$C_2(2m,2n)\mapsto \frac{1}{2}\overline{s^m t^{2n}(t^{-1}dt)}.$$

□

It is clear that the elements

$$(x_+ + x_-)\otimes s^{2m+1}t^{2n},\quad (x_+ - x_-)\otimes s^{2m+1}t^{2n+1},$$

$$\alpha \otimes s^{2m}t^{2n+1}, \quad \overline{s^{2m}t^{2n}(s^{-1}ds)}, \quad \overline{s^{2m}t^{2n}(t^{-1}dt)},$$

for $m, n \in \mathbf{Z}$, form a subalgebra, which we call $\hat{\mathcal{G}}_2$, of the principal toroidal Lie algebra of type A_1 spanned by the elements

$$(x_+ + x_-) \otimes s^{2m+1}t^n, \quad (x_+ - x_-) \otimes s^{2m+1}t^n,$$

$$\alpha \otimes s^{2m}t^n, \quad \overline{s^{2m}t^n(s^{-1}ds)}, \quad \overline{s^{2m}t^n(t^{-1}dt)},$$

for $m, n \in \mathbf{Z}$.

Theorem 3.2.8 also gives the following result

Corollary 3.5.3 The operators

$$\alpha(2m+1, 2n), \quad \alpha(m, 2n+1), \quad C_i(2m, 2n),$$

for $i = 1, 2, m, n \in \mathbf{Z}$, acting on V form a Lie algebra which is isomorphic to the Lie algebra $\hat{\mathcal{G}}_2$ above via the homomorphism ρ_3:

$$\alpha(2m+1, 2n) \mapsto (x_+ + x_-) \otimes s^{2m+1}t^{2n},$$

$$\alpha(2m+1, 2n+1) \mapsto (x_- - x_+) \otimes s^{2m+1}t^{2n+1},$$

$$\alpha(2m, 2n+1) \mapsto \alpha \otimes s^{2m}t^{2n+1},$$

$$C_1(2m, 2n) \mapsto \frac{1}{2}\overline{s^{2m}t^{2n}(s^{-1}ds)},$$

$$C_2(2m, 2n) \mapsto \frac{1}{2}\overline{s^{2m}t^{2n}(t^{-1}dt)},$$

for $m, n \in \mathbf{Z}$.

Chapter 4

Vertex Operator Representation for Toroidal Lie Algebra of Type B_l

§4.1 Introduction

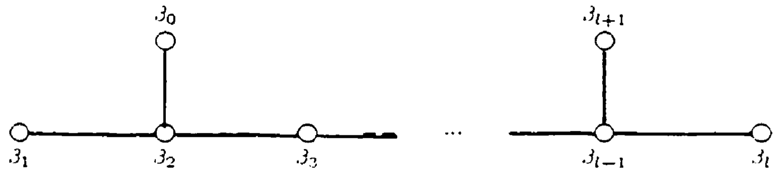
In this chapter we study the vertex operator representations for the toroidal Lie algebra of type B_l ($l \geq 3$). We will give two constructions of this algebra.

In the next section we first set up some notation and recall the root system $\Phi(B_l^{(1)})$ of the affine Kac-Moody algebra $B_l^{(1)}$ ($l \geq 3$). We also recall a presentation for the toroidal Lie algebras of type X_l for $X = A, B, \ldots, G$, which we will make use of. This presentation was discovered in [MRY]. In Section 4.3 we start with an integral lattice Q which indeed contains the affine root lattices $Q(D_{l+1}^{(1)})$ and $Q(B_l^{(1)})$ as sub-lattices. To define the full Fock space V [see (4.3.6)], we define a group algebra $\mathbb{C}[Q]$ of Q with the twisted multiplication $e^{\beta}e^{\gamma} = \epsilon(\beta, \gamma)e^{\beta+\gamma}$ for $\beta, \gamma \in Q$, where the map ϵ: $Q \times Q \rightarrow \{\kappa | \kappa^4 = 1\}$ satisfies the 2-cocycle condition [see Lemma 4.3.1]. We close this section with the main result, Theorem 4.3.6, of this chapter. Section 4.4 is devoted to the proof of Theorem 4.3.6. It follows from the proof of Theorem 4.3.6 that indeed the Fock space V also afford a vertex operator representation of the Clifford algebra $\mathcal{W}$, which is spanned by the elements ω_j ($j \in 2\mathbf{Z} + 1$) with the relation $\omega_i\omega_j + \omega_j\omega_i = -2\delta_{i+j,0}$, ($i, j \in 2\mathbf{Z} + 1$). We summarize this result in Theorem 4.5.2 in Section 4.5. Finally, in Section 4.6 we give the second construction of the toroidal Lie algebra of type B_l [see Theorem 4.6.1]. Here we directly introduce the Clifford algebra $\mathcal{W}$ and its standard irreducible module $\Lambda(\mathcal{W}^-)$ into the picture. In fact, this construction generalizes the Lepowsky-Primc construction of the level one standard module of $B_1^{(1)}$ [LP1] to the toroidal case.

§4.2 Toroidal Lie Algebra of Type B_l

Let $\mathbf{R}^{l+2}$ be the real Euclidean space ($l \geq 3$) with inner product $(\cdot,\cdot)$. Let $\Phi(D_{l+1}^{(1)})$ be a root system of the affine Kac-Moody algebra $D_{l+1}^{(1)}$. We suppose that the form $(\cdot,\cdot)$ is normalized so that $(\alpha,\alpha) = 2$ for all $\alpha \in \Phi(D_{l+1}^{(1)})$ [cf. [K2],[MP]] .

Let $\Pi(D_{l+1}^{(1)}) = \{\beta_i |\ i = 0, 1, \ldots, l+1\}$ be a base of $\Phi(D_{l+1}^{(1)})$. Let $\Gamma(D_{l+1}^{(1)})$ be the Dynkin diagram of $\Phi(D_{l+1}^{(1)})$. Thus $\Gamma(D_{l+1}^{(1)})$ is



Let θ be the diagram automorphism of $D_{l+1}^{(1)}$ of order 2 defined by

(4.2.1) $$\theta(\beta_i) = \beta_i, \quad \theta(\beta_l) = \beta_{l+1}, \quad \text{for } i = 0, 1, \ldots, l-1.$$

Let $\delta = \sum_{i=0}^{l+1} n_i \beta_i$ be the null root with $n_0 = 1$. Let d be a symbol. We form a vector space

(4.2.2) $$H_0 = \left(\dot{+}_{i=0}^{l+1} \mathbf{C}\beta_i\right) \dot{+} \mathbf{C}d,$$

with the extended symmetric bilinear form $(\cdot,\cdot) : \ H_0 \times H_0 \to \mathbf{C}$ so that

$$(d,\delta) = 1, \quad (d,d) = 0 = (d,\beta_i),$$

for $i = 1, 2, \ldots, l+1$.

It is clear that H_0 contains a nondegenerate integral lattice

(4.2.3) $$\Gamma = \left(\dot{+}_{i=0}^{l+1} \mathbf{Z}\alpha_i\right) \dot{+} \mathbf{Z}d,$$

where

$$\alpha_i = \frac{1}{2}(\beta_i + \theta(\beta_i)), \quad \text{for } i = 0, 1, \ldots, l,$$

$$\alpha_{l+1} = \frac{1}{2}(\beta_l - \theta(\beta_l)).$$

Moreover, Γ contains two sublattices

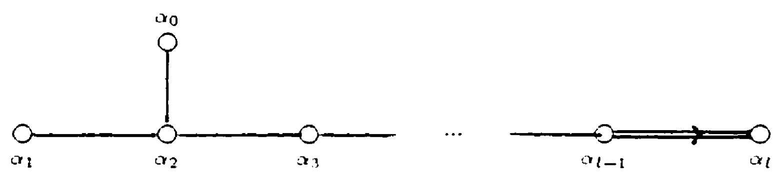
$$Q(D_{l+1}^{(1)}) = \oplus_{i=0}^{l+1}\mathbf{Z}\beta_i,$$

and

$$Q(B_l^{(1)}) = \oplus_{i=0}^{l}\mathbf{Z}\alpha_i,$$

which are indeed the root lattices of the affine Lie algebras $D_{l+1}^{(1)}$ and $B_l^{(1)}$ respectively. Indeed, we have [cf. [K2] or [MP]]

Lemma 4.2.1 The set $\Phi(B_l^{(1)}) := \{\frac{1}{2}(\beta + \theta(\beta)) \mid \beta \in \Phi(D_{l+1}^{(1)})\}$ form a root system of the affine Lie algebra $B_l^{(1)}$ with base $\{\alpha_i \mid i = 0, 1, \ldots, l\}$. Moreover, the corresponding Dynkin diagram of $\Phi(B_l^{(1)})$ is



□

Set $\alpha_j^\vee = \frac{2\alpha_j}{(\alpha_j, \alpha_j)}$ for $0 \le j \le l$, and $A_{ij} = (\alpha_i, \alpha_j^\vee)$ for $i, j = 0, 1, 2, \ldots, l$. Then $A = (A_{ij})_{i,j=0}^{l}$ is an indecomposable Cartan matrix of affine type $B_l^{(1)}$. We denote by Φ_L and Φ_S the sets of all long and short roots respectively in the root system $\Phi(B_l^{(1)})$. Then, we have the following properties:

(a). If $\alpha, \beta \in \Phi_L$, then $|(\alpha, \beta)| \le 2$, and moreover if $(\alpha, \beta) \ge 0$, $\alpha + \beta \notin \Phi$; if $(\alpha, \beta) = -1$, $\alpha + \beta \in \Phi_L$; if $(\alpha, \beta) = -2$, $\beta = -\alpha$.

(b). If $\alpha \in \Phi_L$, $\beta \in \Phi_S$, then $|(\alpha, \beta)| \le 1$. Moreover, if $(\alpha, \beta) \ge 0$, $\alpha + \beta \notin \Phi$; if $(\alpha, \beta) = -1$, $\alpha + \beta \in \Phi_S$.

(c). If $\alpha, \beta \in \Phi_S$, then $|(\alpha, \beta)| \le 1$. Moreover, if $(\alpha, \beta) > 0$, $\alpha + \beta \notin \Phi$; if $(\alpha, \beta) = 0$, then either $\alpha + \beta \in \Phi_L$ or $\alpha + \beta \notin \Phi$; if $(\alpha, \beta) = -1$, $\beta = -\alpha$.

Generally, suppose $A = (A_{ij})_{i,j=0}^{l}$ is an indecomposable Cartan matrix of affine type $X_l^{(1)}$ ($X = A, B, \ldots, G$), where $\dot{A} = (A_{ij})_{i,j=1}^{l}$ is of finite type X_l. Let $\mathcal{G} = \mathcal{G}(A)$, and $\dot{\mathcal{G}} = \mathcal{G}(\dot{A})$ be the corresponding affine and finite Lie algebras respectively.

Let $\mathcal{L}(A)$ be the Lie algebra over $\mathbf{C}$ with the following presentation:

Generators: c, $\alpha_i^\vee(k)$, $x_k(\pm\alpha_i)$, where $i = 0, 1, \ldots, l$, $k \in \mathbf{Z}$.

Relations:

(R1). $[c, \alpha_i^\vee(k)] = 0 = [c, x_k(\pm\alpha_i)]$.

(R2). $[\alpha_i^\vee(k), \alpha_j^\vee(m)] = k(\alpha_i^\vee, \alpha_j^\vee)\delta_{k+m,0}c$.

(R3). $[\alpha_i^\vee(k), x_m(\pm\alpha_j)] = \pm(\alpha_i^\vee, \alpha_j)x_{m+k}(\pm\alpha_j)$.

(R4). $[x_m(\alpha_i), x_n(-\alpha_j)] = -\delta_{ij}\left\{\alpha_i^\vee(m+n) + \frac{2m\delta_{m+n,0}}{(\alpha_i,\alpha_j)}c\right\}$.

(R5). $[x_m(\alpha_i), x_n(\alpha_i)] = 0 = [x_m(-\alpha_i), x_n(-\alpha_i)]$.

$$\left.\begin{array}{r}(\mathrm{ad}x_0(\alpha_i))^{-A_{ji}+1}x_m(\alpha_j) = 0\\ (\mathrm{ad}x_0(-\alpha_i))^{-A_{ji}+1}x_m(-\alpha_j) = 0\end{array}\right\}\ i \neq j.$$

for all $i, j = 0, 1, \ldots, l$, and $k, m, n \in \mathbf{Z}$.

Then, we have

Proposition 4.2.2 [see [MRY]] The Lie algebra $\mathcal{L}(A)$ is the universal central extension of $\dot{\mathcal{G}} \otimes \mathbf{C}[s, s^{-1}, t, t^{-1}]$. That is, $\mathcal{L}(A)$ is isomorphic to the toroidal Lie algebra of type X_l (in two variables), where $X = A, B, \ldots, G$.

□

As before, we describe the relations (R3)-(R5) in terms of formal power series identities.

Proposition 4.2.3 Let $z, z_1, z_2, \ldots$ be formal variables. Let

$$\alpha_i^\vee(z) = \sum_{k\in\mathbf{Z}} \alpha_i^\vee(k)z^{-k},$$

$$X(\pm\alpha_i, z) = \sum_{k\in\mathbf{Z}} x_k(\pm\alpha_i)z^{-k}.$$

Then the relations (R3)-(R5) can be described by the following formal power series identities

$$(4.2.4)\qquad [\alpha_i^\vee(k), X(\pm\alpha_j, z)] = \pm(\alpha_i^\vee, \alpha_j)z^k X(\pm\alpha_j, z).$$

$$(4.2.5)\qquad [X(\alpha_i, z_1), X(-\alpha_j, z_2)] = -\delta_{ij}\left(\alpha_i^\vee(z_2)\delta(\frac{z_2}{z_1}) + \frac{2}{(\alpha_i, \alpha_j)}(D\delta)(\frac{z_2}{z_1})c\right).$$

$$(4.2.6) \qquad [X(\alpha_i, z_1), X(\alpha_i, z_2)] = 0 = [X(-\alpha_i, z_1), X(-\alpha_i, z_2)],$$

$$\left.\begin{array}{r} [X(\alpha_i, z_p), \cdots, X(\alpha_i, z_2), X(\alpha_j, z_1)] = 0 \\ [X(-\alpha_i, z_p), \cdots, X(-\alpha_i, z_2), X(-\alpha_j, z_1)] = 0 \end{array}\right\} \quad i \neq j,$$

where $p = -A_{ji} + 2$.

Proof. The relation (R3) follows from the identity (4.2.4) by equating the coefficients of z^{-m}, while the relation (R4) follows from the identity (4.2.5) by equating the coefficients of $z_1^{-m} z_2^{-n}$. Finally, the relation (R5) follows from the identities in (4.2.6) by equating the coefficients of $z_1^{-m} z_2^{-n}$ for the first part of (4.2.6) and the coefficients of z_1^{-m} for the second part.

□

§4.3 First Construction of the Toroidal Lie Algebra of Type B_l

We define a Lie algebra

$$(4.3.1) \qquad \check{\mathcal{H}}_0 = (\dot{+}_{k\in\mathbf{Z}} H_0(k)) \dot{+} \mathbf{C}c,$$

for a collection of linear copies $H_0(k)$ of H_0[see (4.2.2)] for $k \in \mathbf{Z}$, with the Lie product

$$(4.3.2) \qquad [\alpha(m), \beta(n)] = m(\alpha, \beta)\delta_{m+n,0}c,$$

$$[\check{\mathcal{H}}_0, c] = 0,$$

for $\alpha, \beta \in H_0$, $m, n \in \mathbf{Z}$.

It is clear that $\check{\mathcal{H}}_0$ contains a Heisenberg subalgebra of the form

$$(4.3.3) \qquad \hat{\mathcal{H}}_0 = \left(\dot{+}_{k\in\mathbf{Z}\setminus\{0\}} H_0(k)\right) \dot{+} \mathbf{C}c.$$

Let $\mathcal{S}(\hat{\mathcal{H}}_0^-)$ be the standard irreducible $\hat{\mathcal{H}}_0$-module with the action of $\hat{\mathcal{H}}_0$ on the symmetric algebra $\mathcal{S}(\hat{\mathcal{H}}_0)$ defined by setting c to act as 1, $a(-m)$ to act as multiplication and $a(m)$ to act as partial differential operator for which

$$a(m).b(-n) = m(\alpha, \beta)\delta_{m-n,0},$$

where $m, n \in \mathbf{Z}_+$, $a, b \in H_0$.

Now we want to define an integral lattice Q so that it will contain the affine root lattices of $B_l^{(1)}$ and $D_{l+1}^{(1)}$ as sublattices. With this lattice Q we will define the full Fock space V (see (4.3.6)], which affords representations for the toroidal Lie algebras of both type B_l and type D_{l+1} by vertex operators. For this purpose we let $Q = \dotplus_{i=0}^{l+1}\mathbf{Z}\alpha_i \subset \Gamma$. We define a map $\epsilon: \ Q \times Q \to \{\kappa | \ \kappa^4 = 1\}$ by setting

$$\epsilon(\alpha_i, \alpha_j) = \begin{cases} 1, & \text{if } 0 \le i < j \le l, \\ (-1)^{(\alpha_i,\alpha_i)-1}, & \text{if } 0 \le i = j \le l, \\ (-1)^{(\alpha_i,\alpha_j)}, & \text{if } 0 \le j < i \le l, \end{cases}$$

$$\epsilon(\alpha_i, \alpha_{l+1}) = 1 = \epsilon(\alpha_{l+1}, \alpha_i), \qquad \text{for } i = 0, 1, \ldots, l-1,$$

$$\epsilon(\alpha_l, \alpha_{l+1}) = \sqrt{-1} = \epsilon(\alpha_{l+1}, \alpha_l), \qquad \epsilon(\alpha_{l+1}, \alpha_{l+1}) = 1,$$

and such that

$$(4.3.4) \qquad \epsilon(\sum_{i=0}^{l+1} m_i\alpha_i, \sum_{i=0}^{l+1} n_i\alpha_i) = \prod_{i=0}^{l+1} (\epsilon(\alpha_i, \alpha_j))^{m_i n_j},$$

for $m_i, n_j \in \mathbf{Z}$.

Lemma 4.3.1 The map ϵ: $\ Q \times Q \to \{\kappa | \ \kappa^4 = 1\}$ is a two-cocycle on the integral lattice Q. That is

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma),$$

for $\alpha, \beta, \gamma \in Q$.

In particular, if we restrict ϵ on the root lattice $Q(D_{l+1}^{(1)})$, then ϵ gives the usual 2-cocycle ϵ: $\ Q(D_{l+1}^{(1)}) \times Q(D_{l+1}^{(1)}) \to \{\pm 1\}$ of the simply-laced affine root lattice $Q(D_{l+1}^{(1)})$ [cf. [FLM], [MRY]]. That is

$$\epsilon(\beta_i, \beta_i) = (-1)^{\frac{(\beta_i,\beta_i)}{2}}, \qquad \text{for } 0 \le i \le l+1,$$

$$\epsilon(\beta_i, \beta_j) = \begin{cases} 1, & \text{if } 0 \le i < j \le l+1, \\ (-1)^{(\beta_i,\beta_j)}, & \text{if } 0 \le j < i \le l+1, \end{cases}$$

and

$$\epsilon(\sum_{i=0}^{l+1} m_i\beta_i, \sum_{i=0}^{l+1} n_i\beta_i) = \prod_{i,j=0}^{l+1} (\epsilon(\beta_i, \beta_j))^{m_i n_j},$$

for $m_i, n_j \in \mathbf{Z}$.[cf. (4.2.3)]

$\square$

Corollary 4.3.2 If we restrict the map ϵ on the root lattice $Q(B_l^{(1)})$, then ϵ defines a 2-cocycle map: $Q(B_l^{(1)}) \times Q(B_l^{(1)}) \to \{\pm 1\}$ for which

$$\epsilon(\alpha_i, \alpha_j) = \begin{cases} 1, & \text{if } 0 \leq i < j \leq l, \\ (-1)^{(\alpha_i, \alpha_i) - 1}, & \text{if } 0 \leq i = j \leq l, \\ (-1)^{(\alpha_i, \alpha_j)}, & \text{if } 0 \leq j < i \leq l, \end{cases}$$

and

$$\epsilon(\sum_{i=0}^{l} m_i \alpha_i, \sum_{i=0}^{l} n_i \alpha_i) = \prod_{i,j=0}^{l} (\epsilon(\alpha_i, \alpha_j))^{m_i n_j},$$

for $m_i, n_j \in \mathbf{Z}$.

In particular, we have

$$\epsilon(\alpha_i, \pm\alpha_j)\epsilon(\pm\alpha_j, \alpha_i) = \begin{cases} 1, & \text{if } i = j, \\ (-1)^{(\alpha_i, \alpha_j)}, & \text{if } i \neq j, \end{cases}$$

for $i, j = 0, 1, \ldots, l$.

$\square$

As usual, we form a group algebra $\mathbf{C}[Q]$ with base elements of the form e^{γ}, $\gamma \in Q$, and the twisted product

$$e^{\gamma} e^{\tau} = \epsilon(\gamma, \tau) e^{\gamma + \tau}, \tag{4.3.5}$$

for $\gamma, \tau \in Q$.

It is clear that $\mathbf{C}[Q]$ contains two subalgebras $\mathbf{C}[Q(D_{l+1}^{(1)})]$ and $\mathbf{C}[Q(B_l^{(1)})]$.

Define the Fock space

$$V := \mathbf{C}[Q] \otimes \mathcal{S}(\hat{\mathcal{H}}_0^-), \tag{4.3.6}$$

and extend the action of $\hat{\mathcal{H}}_0$ to the space V by defining

$$a(m).e^{\gamma} \otimes u = e^{\gamma} \otimes a(m).u,$$

$$a(0).e^{\gamma} \otimes u = (a, \gamma) e^{\gamma} \otimes u,$$

for $a \in H_0$, $m \in \mathbf{Z}_+$, and $e^{\gamma} \otimes u \in V$.

Remark The Fock space V contains the subspace $V(D_{l+1}^{(1)}) := \mathbb{C}[Q(D_{l+1}^{(1)})] \otimes S(\hat{\mathcal{H}}_0^-)$, which is the Fock space used in [MRY].

For $\beta \in Q$, we define the vertex operator

$$Y(\beta, z) = z^{\frac{(\beta,\beta)}{2}} e^{\beta} z^{\beta} E^-(\beta, z) E^+(\beta, z), \tag{4.3.7}$$

in $(\mathrm{End})[[z^{\frac{1}{2}}, z^{-\frac{1}{2}}]]$, where

$$E^{\pm}(\beta, z) = \exp\left(-\sum_{n\in\mathbf{Z}_{\pm}} \frac{\beta(n)}{n} z^{-n}\right).$$

and the operator z^{β} is defined by

$$z^{\beta}.e^{\gamma} \otimes u = z^{(\beta,\gamma)} e^{\gamma} \otimes u, \tag{4.3.8}$$

for $e^{\gamma} \otimes u \in V$.

Formally, we may expand the vertex operator $Y(\beta, z)$ into formal power series in $z^{\frac{1}{2}}$,

$$Y(\beta, z) = \sum_{n\in\frac{1}{2}\mathbf{Z}} y_m(\beta) z^{-m},$$

for $\beta \in Q$.

Recall that $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}, \alpha_l + \alpha_{l+1}, \alpha_l - \alpha_{l+1}$ form a base of the root system $\Phi(D_{l+1}^{(1)})$ of the affine Lie algebra $D_{l+1}^{(1)}$ [see (4.2.3)]. Therefore, from [MRY], we have

Proposition 4.3.3 The moment operators $y_m(\alpha_l+\alpha_{l+1})$, $y_m(\alpha_l-\alpha_{l+1})$ and $y_m(\alpha_i)$ for $i = 0, 1, \ldots, l-1$, $m \in \mathbf{Z}$, on V (also on $V(D_{l+1}^{(1)})$), generate a Lie algebra which is isomorphic to the toroidal Lie algebra of type D_{l+1} (in two variables) for $l \geq 3$.

□

In order to give a representation of the toroidal Lie algebra of type B_l on the Fock space V, we first study some properties of the vertex operators $Y(\beta, z)$ for $\beta \in Q$.

Lemma 4.3.4 For $\alpha, \beta \in Q$, we have

$$E^+(\alpha, z) E^-(\beta, w) = E^-(\beta, w) E^+(\alpha, z)(1 - \frac{w}{z})^{(\alpha,\beta)}$$

in $(\mathrm{End})[[z^{-1}, w]]$, and

$$z^{\alpha}\epsilon^{\beta} = z^{(\alpha,\beta)}\epsilon^{\beta}z^{\alpha}.$$

Proof. It is standard to prove the above identities by applying (4.3.2), (4.3.5), (4.3.8) and the formal rule:

$$\epsilon^{X}\epsilon^{Y} = \epsilon^{Y}\epsilon^{X}\epsilon^{[X,Y]},$$

if $[X, Y]$ commutes with X and Y.

□

For convenience, we adopt the notation of normal ordering rules defined in Section 1.7 [cf. [FLM]], but here allowing the indices i, j to range through $\frac{1}{2}\mathbb{Z}$. We also define normal ordering for noncommuting expressions involving ϵ^{α} and z^{α} by

$$(4.3.9) \qquad : z^{\alpha}\epsilon^{\beta} :=: \epsilon^{\beta}z^{\alpha} := z^{\frac{(\alpha,\beta)}{2}}\epsilon^{\beta}z^{\alpha},$$

$$: \epsilon^{\alpha}\epsilon^{\beta} :=: \epsilon^{\beta}\epsilon^{\alpha} := \epsilon^{\alpha+\beta},$$

for $\alpha, \beta \in Q$.

Now, we can rewrite the vertex operator $Y(\beta, z)$ [see (4.3.7)] as follows

$$(4.3.10) \qquad Y(\beta, z) =: \epsilon^{\beta}z^{\beta}E^{-}(\beta, z)E^{+}(\beta, z) : .$$

We also define

$$(4.3.11) \qquad : Y(\alpha, z)Y(\beta, w) :$$

$$= z^{\frac{(\alpha,\alpha+\beta)}{2}}w^{\frac{(\beta,\alpha+\beta)}{2}}\epsilon^{\alpha+\beta}z^{\alpha}w^{\beta}E^{-}(\alpha, z)E^{-}(\beta, w)E^{+}(\alpha, z)E^{+}(\beta, w).$$

Under this normal ordering, we have the commutative relation

$$(4.3.12) \qquad : Y(\alpha, z)Y(\beta, w) :=: Y(\beta, w)Y(\alpha, z) :$$

and

$$(4.3.13) \qquad \lim_{w\to z} : Y(\alpha, z)Y(\beta, w) :$$

$$= z^{\frac{(\alpha,\alpha+\beta)}{2}}z^{\frac{(\beta,\alpha+\beta)}{2}}\epsilon^{\alpha+\beta}z^{\alpha}z^{\beta}E^{-}(\alpha, z)E^{-}(\beta, z)E^{+}(\alpha, z)E^{+}(\beta, z)$$

$$= z^{\frac{(\alpha+\beta,\alpha+\beta)}{2}} \epsilon^{\alpha+\beta} z^{\alpha+\beta} E^-(\alpha+\beta,z)E^+(\alpha+\beta,z)$$

$$= Y(\alpha+\beta,z).$$

From (4.3.11) and Lemma 4.3.4, we conclude

Proposition 4.3.5 For $\alpha,\beta \in Q$, we have

$$Y(\alpha,z)Y(\beta,w) = \epsilon(\alpha,\beta) : Y(\alpha,z)Y(\beta,w) : (\frac{w}{z})^{-\frac{(\alpha,\beta)}{2}}(1-\frac{w}{z})^{(\alpha,\beta)}.$$

□

In what follows, we want to show that the Fock space V also afford a representation for the toroidal Lie algebra of type B_l. For this purpose, we define the vertex operators, for $\alpha_i \in \Phi(B_l^{(1)}) \subset Q$

$$X(\alpha_i,z) = \begin{cases} Y(\alpha_i,z), & \text{if } 0 \le i \le l-1, \\ \sqrt{-1}\,(Y(\alpha_l+\alpha_{l+1},z)+Y(\alpha_l-\alpha_{l+1},z)), & \text{if } i = l, \end{cases}$$

for $i = 0,1,\ldots,l$.

As before, we may formally expand $X(\alpha_i,z)$ into power series to obtain

$$X(\alpha_i,z) = \sum_{n\in\mathbf{Z}} x_m(\alpha_i)z^{-m},$$

for $i = 0,1,\ldots,l$.

Recall that $\beta_l = \alpha_l+\alpha_{l+1}$, $\beta_{l+1} = \alpha_l-\alpha_{l+1}$ and $\beta_i = \alpha_i$ for $0 \le i \le l-1$. We see that the moments operators $x_m(\alpha_i)$ for $0 \le i \le l$ act also on $V(D_{l+1}^{(1)})$.

Now, we state the main result of this chapter

Theorem 4.3.6 The moment operators $x_m(\alpha_i)$, $m \in \mathbf{Z}$, $i = 0,1,\ldots,l$, acting on V (also on $V(D_{l+1}^{(1)})$), generate a Lie algebra which is isomorphic to the toroidal Lie algebra (in two variables) of type B_l for $l \ge 3$.

□

§4.4 Proof of Theorem 4.3.6

Recall that $\{\alpha_0,\alpha_1,\ldots,\alpha_l\}$ form a base of the root system $\Phi(B_l^{(1)})$ [see Lemma 4.2.1], and α_l is the only short simple root of $B_l^{(1)}$. Now, we first study the structure of the vertex operator $X(\alpha_l,z)$.

By (4.3.7), (4.3.5), and note that $(\alpha_l, \alpha_{l+1}) = 0$, we have

$$(4.4.1) \qquad X(\alpha_l, z) = \sqrt{-1}(Y(\alpha_l + \alpha_{l+1}, z) + Y(\alpha_l - \alpha_{l+1})$$

$$= \sqrt{-1} z^{\frac{(\alpha_l+\alpha_{l+1}, \alpha_l+\alpha_{l+1})}{2}} \epsilon^{\alpha_l+\alpha_{l+1}} z^{\alpha_l} z^{\alpha_{l+1}} E^-(\alpha_l, z) E^-(\alpha_{l+1}, z)$$

$$\cdot E^+(\alpha_l, z) E^+(\alpha_{l+1}, z)$$

$$+ \sqrt{-1} z^{\frac{(\alpha_l-\alpha_{l+1}, \alpha_l-\alpha_{l+1})}{2}} \epsilon^{\alpha_l-\alpha_{l+1}} z^{\alpha_l} z^{-\alpha_{l+1}} E^-(\alpha_l, z) E^-(-\alpha_{l+1}, z)$$

$$\cdot E^+(\alpha_l, z) E^+(-\alpha_{l+1}, z)$$

$$= \sqrt{-1} z (\epsilon(\alpha_l, \alpha_{l+1}))^{-1} \epsilon^{\alpha_l} \epsilon^{\alpha_{l+1}} z^{\alpha_l} z^{\alpha_{l+1}} E^-(\alpha_l, z) E^-(\alpha_{l+1}, z)$$

$$\cdot E^+(\alpha_l, z) E^+(\alpha_{l+1}, z)$$

$$+ \sqrt{-1} z (\epsilon(\alpha_l, -\alpha_{l+1}))^{-1} \epsilon^{\alpha_l} \epsilon^{-\alpha_{l+1}} z^{\alpha_l} z^{-\alpha_{l+1}} E^-(\alpha_l, z) E^-(-\alpha_{l+1}, z)$$

$$\cdot E^+(\alpha_l, z) E^+(-\alpha_{l+1}, z)$$

$$= z \epsilon^{\alpha_l} \epsilon^{\alpha_{l+1}} z^{\alpha_l} z^{\alpha_{l+1}} E^-(\alpha_l, z) E^-(\alpha_{l+1}, z) E^+(\alpha_l, z) E^+(\alpha_{l+1}, z)$$

$$- z \epsilon^{\alpha_l} \epsilon^{-\alpha_{l+1}} z^{\alpha_l} z^{-\alpha_{l+1}} E^-(\alpha_l, z) E^-(-\alpha_{l+1}, z) E^+(\alpha_l, z) E^+(-\alpha_{l+1}, z)$$

$$= \epsilon^{\alpha_l} z^{\alpha_l} E^-(\alpha_l, z) \{ z \epsilon^{\alpha_{l+1}} z^{\alpha_{l+1}} E^-(\alpha_{l+1}, z) E^+(\alpha_{l+1}, z)$$

$$- z \epsilon^{-\alpha_{l+1}} z^{-\alpha_{l+1}} E^-(-\alpha_{l+1}, z) E^+(-\alpha_{l+1}, z) \} E^+(\alpha_l, z)$$

$$= \epsilon^{\alpha_l} z^{\alpha_l} E^-(\alpha_l, z) W(z) E^+(\alpha_l, z),$$

where

$$(4.4.2) \qquad W(z) = z \epsilon^{\alpha_{l+1}} z^{\alpha_{l+1}} E^-(\alpha_{l+1}, z) E^+(\alpha_{l+1}, z)$$

$$- z \epsilon^{-\alpha_{l+1}} z^{-\alpha_{l+1}} E^-(-\alpha_{l+1}, z) E^+(-\alpha_{l+1}, z)$$

in $(\mathrm{End} V)[[z, z^{-1}]]$.

Let ω_{2j+1} for $j \in \mathbf{Z}$ be the operators, on V, defined by the expansion of $W(z)$, that is

$$(4.4.3) \qquad W(z) := \sum_{j \in \mathbf{Z}} \omega_{2j+1} z^{-j}.$$

Then, we have

Lemma 4.4.1 The operators $\{\omega_{2j+1}\}_{j\in\mathbf{Z}}$, acting on V, satisfy the following relation

$$\omega_i\omega_j + \omega_j\omega_i = -2\delta_{i+j,0}, \tag{4.4.4}$$

for $i, j \in 2\mathbf{Z}+1$. Hence $\mathcal{W} := \mathrm{span}_{\mathbf{C}}\{\omega_j, j \in 2\mathbf{Z}+1\}$ forms a Clifford algebra.

Proof. Recalling (4.3.7), we write $W(z)$ as follows

$$W(z) = z^{\frac{1}{2}}(Y(\alpha_{l+1}, z) - Y(-\alpha_{l+1}, z)).$$

Thus, by (4.3.12), Proposition 4.3.5, and the fact: $\epsilon(\alpha_{l+1}, \alpha_{l+1}) = 1$, we have

$$
\begin{aligned}
&W(z_1)W(z_2)\\
&= z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}(Y(\alpha_{l+1}, z_1) - Y(-\alpha_{l+1}, z_1))(Y(\alpha_{l+1}, z_2) - Y(-\alpha_{l+1}, z_2))\\
&= z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}\{Y(\alpha_{l+1}, z_1)Y(\alpha_{l+1}, z_2) + Y(-\alpha_{l+1}, z_1)Y(-\alpha_{l+1}, z_2)\}\\
&\quad - z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}\{Y(\alpha_{l+1}, z_1)Y(-\alpha_{l+1}, z_2) + Y(-\alpha_{l+1}, z_1)Y(\alpha_{l+1}, z_2)\}\\
&= z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}\{: Y(\alpha_{l+1}, z_1)Y(\alpha_{l+1}, z_2) : (\frac{z_2}{z_1})^{-\frac{\langle\alpha_{l+1},\alpha_{l+1}\rangle}{2}}(1 - \frac{z_2}{z_1})^{\langle\alpha_{l+1},\alpha_{l+1}\rangle}\\
&\quad + : Y(-\alpha_{l+1}, z_1)Y(-\alpha_{l+1}, z_2) : (\frac{z_2}{z_1})^{-\frac{\langle-\alpha_{l+1},-\alpha_{l+1}\rangle}{2}}(1 - \frac{z_2}{z_1})^{\langle-\alpha_{l+1},-\alpha_{l+1}\rangle}\}\\
&\quad - z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}\{: Y(\alpha_{l+1}, z_1)Y(-\alpha_{l+1}, z_2) : (\frac{z_2}{z_1})^{-\frac{\langle\alpha_{l+1},-\alpha_{l+1}\rangle}{2}}(1 - \frac{z_2}{z_1})^{\langle\alpha_{l+1},-\alpha_{l+1}\rangle}\\
&\quad + : Y(-\alpha_{l+1}, z_1)Y(\alpha_{l+1}, z_2) : (\frac{z_2}{z_1})^{-\frac{\langle-\alpha_{l+1},\alpha_{l+1}\rangle}{2}}(1 - \frac{z_2}{z_1})^{\langle-\alpha_{l+1},\alpha_{l+1}\rangle}\}\\
&= z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}\{: Y(\alpha_{l+1}, z_1)Y(\alpha_{l+1}, z_2) : (\frac{z_2}{z_1})^{-\frac{1}{2}}(1 - \frac{z_2}{z_1})\\
&\quad + : Y(-\alpha_{l+1}, z_1)Y(-\alpha_{l+1}, z_2) : (\frac{z_2}{z_1})^{-\frac{1}{2}}(1 - \frac{z_2}{z_1})\}\\
&\quad - z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}\{: Y(\alpha_{l+1}, z_1)Y(-\alpha_{l+1}, z_2) : (\frac{z_2}{z_1})^{\frac{1}{2}}(1 - \frac{z_2}{z_1})^{-1}\\
&\quad + : Y(-\alpha_{l+1}, z_1)Y(\alpha_{l+1}, z_2) : (\frac{z_2}{z_1})^{\frac{1}{2}}(1 - \frac{z_2}{z_1})^{-1}\}\\
&= \{: Y(\alpha_{l+1}, z_1)Y(\alpha_{l+1}, z_2) : + : Y(-\alpha_{l+1}, z_1)Y(-\alpha_{l+1}, z_2) :\}(z_1 - z_2)
\end{aligned}
$$

$$-\{:Y(\alpha_{l+1},z_1)Y(-\alpha_{l+1},z_2):+:Y(-\alpha_{l+1},z_1)Y(\alpha_{l+1},z_2):\}z_2(1-\frac{z_2}{z_1})^{-1}.$$

Therefore, by symmetry, we obtain

$$\begin{aligned} &W(z_1)W(z_2)+W(z_2)W(z_1)\\ &=-\{:Y(\alpha_{l+1},z_1)Y(-\alpha_{l+1},z_2):+:Y(-\alpha_{l+1},z_1)Y(\alpha_{l+1},z_2):\}\\ &\cdot\{z_2(1-\frac{z_2}{z_1})^{-1}+z_1(1-\frac{z_1}{z_2})^{-1}\},\end{aligned}$$

where

$$\begin{aligned} &z_2(1-\frac{z_2}{z_1})^{-1}+z_1(1-\frac{z_1}{z_2})^{-1}\\ &=z_2\left((1-\frac{z_2}{z_1})^{-1}+\frac{z_1}{z_2}(1-\frac{z_1}{z_2})^{-1}\right)=z_2\delta(\frac{z_2}{z_1}).\end{aligned}$$

Therefore, by Lemma 1.5.1 and (4.3.13), we obtain

$$\begin{aligned} &W(z_1)W(z_2)+W(z_2)W(z_1)\\ &=-\{:Y(\alpha_{l+1},z_1)Y(-\alpha_{l+1},z_2):+:Y(-\alpha_{l+1},z_1)Y(\alpha_{l+1},z_2):\}z_2\delta(\frac{z_2}{z_1})\\ &=-\lim_{z_1\to z_2}\{:Y(\alpha_{l+1},z_1)Y(-\alpha_{l+1},z_2):+:Y(-\alpha_{l+1},z_1)Y(\alpha_{l+1},z_2):\}z_2\delta(\frac{z_2}{z_1})\\ &=-2Y(0,z_2)z_2\delta(\frac{z_2}{z_1})=-2z_2\delta(\frac{z_2}{z_1}).\end{aligned}$$

That is

$$W(z_1)W(z_2)+W(z_2)W(z_1)=-2z_2\delta(\frac{z_2}{z_1}).$$

Finally, if we write both sides of the above identity into power series in z_1 and z_2, and equate the coefficients of $z_1^{-i}z_2^{-j}$ for $i,j\in\mathbf{Z}$, we obtain

$$\omega_i\omega_j+\omega_j\omega_i=-2\delta_{i+j,0},\tag{4.4.5}$$

for $i,j\in 2\mathbf{Z}+1$, as required.

□

Before we start to prove Theorem 4.3.6, we state some basic identities. First we note that $(\alpha, \alpha_{l+1}) = 0$ for all $\alpha \in \Phi(B_l^{(1)})$, thus we can easily check the following relations, for $\alpha \in \Phi(B_l^{(1)})$

$$(4.4.6) \qquad [E^{\pm}(\alpha, z_1), W(z_2)] = 0,$$

$$[\epsilon^{\alpha}, W(z)] = 0 = [z^{\alpha}, W(z)].$$

Therefore, by the definition of $X(\alpha_i, z)$, and (4.4.1), we have

$$X(\alpha_i, z) = z^{\frac{(\alpha_i, \alpha_i)}{2}} \epsilon^{\alpha_i} z^{\alpha_i} E^-(\alpha_i, z) E^+(\alpha_i, z),$$

for $i = 0, 1, \ldots, l-1$, and

$$X(\alpha_l, z) = \epsilon^{\alpha_l} z^{\alpha_l} E^-(\alpha_l, z) W(z) E^+(\alpha_l, z).$$

For simplicity, we set $\omega_i = \delta_{i,0}$, for $i \in 2\mathbf{Z}$, and set

$$W_{\alpha}(z) = \sum_{m \in \mathbf{Z}} \omega_{2m+(\alpha,\alpha)} z^{-m},$$

for $\alpha \in Q(B_l^{(1)})$. Thus, if $(\alpha, \alpha) = 1$, we have $W_{\alpha}(z) = W(z)$. While if $(\alpha, \alpha) \in 2\mathbf{Z}$, then

$$W_{\alpha}(z) = z^{\frac{(\alpha,\alpha)}{2}}.$$

Therefore, we have

$$X(\alpha_i, z) = \epsilon^{\alpha_i} z^{\alpha_i} E^-(\alpha_i, z) W_{\alpha_i}(z) E^+(\alpha_i, z),$$

for all $i = 0, 1, \ldots, l$.

Lemma 4.4.2 If $\alpha, \beta \in \Phi_L$, then

$$(4.4.7) \qquad [X(\alpha, z_1), X(\beta, z_2)]$$

$$= z_1 z_2 \epsilon^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} E^-(\alpha, z_1) E^-(\beta, z_2) E^+(\alpha, z_1) E^+(\beta, z_2) P(z_1, z_2),$$

where

$$P(z_1, z_2) = \epsilon(\alpha, \beta) z_1^{(\alpha,\beta)} (1 - \frac{z_2}{z_1})^{(\alpha,\beta)} - \epsilon(\beta, \alpha) z_2^{(\alpha,\beta)} (1 - \frac{z_1}{z_2})^{(\alpha,\beta)}.$$

Proof. This follows from Lemma 4.3.4.

□

Lemma 4.4.3 Let $\alpha \in \Phi_L, \beta \in \Phi_S$. Then

$$[X(\alpha, z_1), X(\beta, z_2)] = \epsilon^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} E^-(\alpha, z_1) E^-(\beta, z_2) \tag{4.4.8}$$
$$\cdot \left(\sum_{m \in \mathbf{Z}} \omega_{2m+1} z_2^{-m}\right) E^+(\alpha, z_1) E^+(\beta, z_2) Q(z_1, z_2),$$

where

$$Q(z_1, z_2) = z_1 \left(\epsilon(\alpha, \beta) z_1^{(\alpha,\beta)} (1 - \frac{z_2}{z_1})^{(\alpha,\beta)} - \epsilon(\beta, \alpha) z_2^{(\alpha,\beta)} (1 - \frac{z_1}{z_2})^{(\alpha,\beta)}\right).$$

Proof. Since $\alpha \in \Phi_L, \beta \in \Phi_S$, we have

$$X(\alpha, z_1) = z_1 \epsilon^{\alpha} z_1^{\alpha} E^-(\alpha, z_1) E^+(\alpha, z_1),$$

and

$$X(\beta, z_2) = \epsilon^{\beta} z_2^{\beta} E^-(\beta, z_2) W(z_2) E^+(\beta, z_2). \tag{4.4.9}$$

Thus, by Lemma 4.3.4

$$X(\alpha, z_1) X(\beta, z_2)$$
$$= z_1 \epsilon(\alpha, \beta) \epsilon^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} z_1^{(\alpha,\beta)} E^-(\alpha, z_1) E^-(\beta, z_2) W(z_2)$$
$$\cdot E^+(\alpha, z_1) E^+(\beta, z_2) (1 - \frac{z_2}{z_1})^{(\alpha,\beta)}.$$

Similarly, we have

$$X(\beta, z_2) X(\alpha, z_1)$$
$$= z_1 \epsilon(\beta, \alpha) \epsilon^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} z_2^{(\alpha,\beta)} E^-(\alpha, z_1) E^-(\beta, z_2) W(z_2)$$
$$\cdot E^+(\alpha, z_1) E^+(\beta, z_2) (1 - \frac{z_1}{z_2})^{(\alpha,\beta)}.$$

Therefore, we obtain

$$[X(\alpha, z_1), X(\beta, z_2)]$$
$$= z_1 \epsilon^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} E^-(\alpha, z_1) E^-(\beta, z_2) W(z_2) E^+(\alpha, z_1) E^+(\beta, z_2)$$

$$\cdot\left(\epsilon(\alpha,\beta)z_1^{(\alpha,\beta)}(1-\frac{z_2}{z_1})^{(\alpha,\beta)}-\epsilon(\beta,\alpha)z_2^{(\alpha,\beta)}(1-\frac{z_1}{z_2})^{(\alpha,\beta)}\right).$$

as required.

□

Lemma 4.4.4 Let $\alpha,\beta\in\Phi_S$. Then

$$(4.4.10)\qquad [X(\alpha,z_1),X(\beta,z_2)]$$

$$=\epsilon^{\alpha+\beta}z_1^{\alpha}z_2^{\beta}E^-(\alpha,z_1)E^-(\beta,z_2)R(z_1,z_2)E^+(\alpha,z_1)E^+(\beta,z_2),$$

where

$$R(z_1,z_2)=\epsilon(\alpha,\beta)z_1^{(\alpha,\beta)}(1-\frac{z_2}{z_1})^{(\alpha,\beta)}\left(\sum_{m,n\in\mathbf{Z}}\omega_{2m+1}\omega_{2n+1}z_1^{-m}z_2^{-n}\right)$$

$$-\epsilon(\beta,\alpha)z_2^{(\alpha,\beta)}(1-\frac{z_1}{z_2})^{(\alpha,\beta)}\left(\sum_{m,n\in\mathbf{Z}}\omega_{2n+1}\omega_{2m+1}z_1^{-m}z_2^{-n}\right).$$

Proof. By (4.4.9) and Lemma 4.3.4, we have

$$X(\alpha,z_1)X(\beta,z_2)$$

$$=\epsilon(\alpha,\beta)\epsilon^{\alpha+\beta}z_1^{\alpha}z_2^{\beta}z_1^{(\alpha,\beta)}E^-(\alpha,z_1)E^-(\beta,z_2)\left(\sum_{m,n\in\mathbf{Z}}\omega_{2m+1}\omega_{2n+1}z_1^{-m}z_2^{-n}\right)$$

$$\cdot E^+(\alpha,z_1)E^+(\beta,z_2)(1-\frac{z_2}{z_1})^{(\alpha,\beta)}.$$

Therefore, by symmetry, this gives

$$[X(\alpha,z_1),X(\beta,z_2)]=\epsilon^{\alpha+\beta}z_1^{\alpha}z_2^{\beta}E^-(\alpha,z_1)E^-(\beta,z_2)$$

$$\cdot(\epsilon(\alpha,\beta)z_1^{(\alpha,\beta)}(1-\frac{z_2}{z_1})^{(\alpha,\beta)}\sum_{m,n\in\mathbf{Z}}\omega_{2m+1}\omega_{2n+1}z_1^{-m}z_2^{-n}$$

$$-\epsilon(\beta,\alpha)z_2^{(\alpha,\beta)}(1-\frac{z_1}{z_2})^{(\alpha,\beta)}\sum_{m,n\in\mathbf{Z}}\omega_{2n+1}\omega_{2m+1}z_1^{-m}z_2^{-n})$$

$$\cdot E^+(\alpha,z_1)E^+(\beta,z_2),$$

as required.

□

Proposition 4.4.5 Suppose $\alpha \in \Phi_L$, $\beta \in \Phi$, and satisfy

$$(\alpha, \beta) = -1, \quad \epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = -1.$$

Then

$$[X(\alpha, z_1), X(\beta, z_2)] = \epsilon(\alpha, \beta)X(\alpha + \beta, z_2)\delta(\frac{z_2}{z_1}).$$

Proof. We divide the proof into two cases. First, we assume that $\beta \in \Phi_L$. Then, by Lemma 4.4.2, we have

$$[X(\alpha, z_1), X(\beta, z_2)]$$
$$= z_1 z_2 e^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} E^-(\alpha, z_1)E^-(\beta, z_2)E^+(\alpha, z_1)E^+(\beta, z_2)P(z_1, z_2),$$

where

$$\begin{aligned} P(z_1, z_2) &= \epsilon(\alpha, \beta)z_1^{(\alpha,\beta)}(1 - \frac{z_2}{z_1})^{(\alpha,\beta)} - \epsilon(\beta, \alpha)z_2^{(\alpha,\beta)}(1 - \frac{z_1}{z_2})^{(\alpha,\beta)} \\ &= \epsilon(\alpha, \beta)\left(z_1^{-1}(1 - \frac{z_2}{z_1})^{-1} + z_2^{-1}(1 - \frac{z_1}{z_2})^{-1}\right) \\ &= \epsilon(\alpha, \beta)z_1^{-1}\delta(\frac{z_2}{z_1}). \end{aligned}$$

Therefore, we obtain

$$[X(\alpha, z_1), X(\beta, z_2)]$$
$$\begin{aligned} &= \epsilon(\alpha, \beta)z_2 e^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} E^-(\alpha, z_1)E^-(\beta, z_2)E^+(\alpha, z_1)E^+(\beta, z_2)\delta(\frac{z_2}{z_1}) \\ &= \epsilon(\alpha, \beta)z_2 e^{\alpha+\beta} z_2^{\alpha+\beta} E^-(\alpha + \beta, z_2)E^+(\alpha + \beta, z_2)\delta(\frac{z_2}{z_1}) \\ &= \epsilon(\alpha, \beta)X(\alpha + \beta, z_2)\delta(\frac{z_2}{z_1}), \end{aligned}$$

where we have used the fact that $(\alpha + \beta, \alpha + \beta) = 2$. This completes the proof of case 1.

Now, we assume that $\beta \in \Phi_S$. Then, by Lemma 4.4.3, we have

$$[X(\alpha, z_1), X(\beta, z_2)]$$

$$= e^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} E^-(\alpha, z_1)E^-(\beta, z_2)W(z_2)$$

$$\cdot E^+(\alpha, z_1)E^+(\beta, z_2)Q(z_1, z_2),$$

where

$$Q(z_1, z_2) = z_1 \left(\epsilon(\alpha, \beta) z_1^{(\alpha,\beta)} (1 - \frac{z_2}{z_1})^{(\alpha,\beta)} - \epsilon(\beta, \alpha) z_2^{(\alpha,\beta)} (1 - \frac{z_1}{z_2})^{(\alpha,\beta)} \right)$$

$$= z_1 \epsilon(\alpha, \beta) \left(z_1^{-1} (1 - \frac{z_2}{z_1})^{-1} + z_2^{-1} (1 - \frac{z_1}{z_2})^{-1} \right)$$

$$= \epsilon(\alpha, \beta) \delta(\frac{z_2}{z_1}).$$

Therefore, we obtain

$$[X(\alpha, z_1), X(\beta, z_2)]$$

$$= \epsilon(\alpha, \beta) e^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} E^{-}(\alpha, z_1) E^{-}(\beta, z_2) W(z_2)$$

$$\cdot E^{+}(\alpha, z_1) E^{+}(\beta, z_2) \delta(\frac{z_2}{z_1})$$

$$= \epsilon(\alpha, \beta) e^{\alpha+\beta} z_2^{\alpha+\beta} E^{-}(\alpha + \beta, z_2) W(z_2) E^{+}(\alpha + \beta, z_2) \delta(\frac{z_2}{z_1})$$

$$= \epsilon(\alpha, \beta) X(\alpha + \beta, z_2) \delta(\frac{z_2}{z_1}).$$

where we have used the fact that $(\alpha + \beta, \alpha + \beta) = 1$, and $W_{\alpha+\beta}(z_2) = W(z_2)$. This completes the proof of this proposition.

□

Proposition 4.4.6 Suppose $\alpha, \beta \in \Phi_S$, and satisfy $(\alpha, \beta) = 0$, $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = -1$. Then

$$[X(\alpha, z_1), X(\beta, z_2)] = -2\epsilon(\alpha, \beta) X(\alpha + \beta, z_2) \delta(\frac{z_2}{z_1}).$$

Proof. By Lemma 4.4.4, we have

$$[X(\alpha, z_1), X(\beta, z_2)]$$

$$= e^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} E^{-}(\alpha, z_1) E^{-}(\beta, z_2) R(z_1, z_2) E^{+}(\alpha, z_1) E^{+}(\beta, z_2),$$

where

$$R(z_1, z_2) = \epsilon(\alpha, \beta) z_1^{(\alpha,\beta)} (1 - \frac{z_2}{z_1})^{(\alpha,\beta)} \left(\sum_{m,n \in \mathbf{Z}} \omega_{2m+1} \omega_{2n+1} z_1^{-m} z_2^{-n} \right)$$

$$-\epsilon(\beta, \alpha) z_2^{(\alpha,\beta)} (1 - \frac{z_1}{z_2})^{(\alpha,\beta)} \left(\sum_{m,n \in \mathbf{Z}} \omega_{2n+1} \omega_{2m+1} z_1^{-m} z_2^{-n} \right)$$

$$= \epsilon(\alpha, \beta) \sum_{m,n \in \mathbf{Z}} (\omega_{2m+1}\omega_{2n+1} + \omega_{2n+1}\omega_{2m+1}) z_1^{-m} z_2^{-n}$$

$$= \epsilon(\alpha, \beta) \sum_{m,n \in \mathbf{Z}} (-2\delta_{2(m+n+1),0}) z_1^{-m} z_2^{-n} = -2z_2\epsilon(\alpha, \beta)\delta(\frac{z_2}{z_1}).$$

Therefore, we obtain

$$[X(\alpha, z_1), X(\beta, z_2)]$$

$$= -2z_2\epsilon(\alpha, \beta)\epsilon^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} E^-(\alpha, z_1)E^-(\beta, z_2)E^+(\alpha, z_1)E^+(\beta, z_2)\delta(\frac{z_2}{z_1})$$

$$= -2z_2\epsilon(\alpha, \beta)\epsilon^{\alpha+\beta} z_2^{\alpha+\beta} E^-(\alpha+\beta, z_2)E^+(\alpha+\beta, z_2)\delta(\frac{z_2}{z_1})$$

$$= -2\epsilon(\alpha, \beta)X(\alpha+\beta, z_2)\delta(\frac{z_2}{z_1}),$$

where we have used the fact that $(\alpha + \beta, \alpha + \beta) = 2$.

□

Proposition 4.4.7 Suppose $\alpha \in \Phi_L$, $\beta \in \Phi$, and satisfy $(\alpha, \beta) \geq 0$, $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}$. Then

$$[X(\alpha, z_1), X(\beta, z_2)] = 0.$$

Proof. On account of Lemma 4.4.2 and Lemma 4.4.3, we see that the result of this Proposition follows from the following identity

$$\epsilon(\alpha, \beta) z_1^{(\alpha,\beta)}(1 - \frac{z_2}{z_1})^{(\alpha,\beta)} - \epsilon(\beta, \alpha) z_2^{(\alpha,\beta)}(1 - \frac{z_1}{z_2})^{(\alpha,\beta)}$$

$$= \epsilon(\alpha, \beta)\left((z_1 - z_2)^{(\alpha,\beta)} - (-1)^{(\alpha,\beta)}(z_2 - z_1)^{(\alpha,\beta)}\right) = 0.$$

□

Now, we prove Theorem 4.3.6 by applying Proposition 4.2.2. It is clear that the relations (R1) and (R2) follow from the Heisenberg relation (4.3.2). To check the relation (R3), we are going to prove (4.2.4). That is

$$[\alpha_i^\vee(k), X(\pm\alpha_j, z)] = \pm(\alpha_i^\vee, \alpha_j) z^k X(\pm\alpha_j, z).$$

First, we assume that $k = 0$. We have

$$[\alpha_i^\vee(0), X(\pm\alpha_j, z)]$$

$$= [\alpha_i^\vee(0), \epsilon^{\pm\alpha_j} z^{\pm\alpha_j} E^-(\pm\alpha_j, z) W_{\pm\alpha_j}(z) E^+(\pm\alpha_j, z)]$$

$$= [\alpha_i^\vee(0), \epsilon^{\pm\alpha_j}] z^{\pm\alpha_j} E^-(\pm\alpha_j, z) W_{\pm\alpha_j}(z) E^+(\pm\alpha_j, z),$$

where

$$[\alpha_i^\vee(0), \epsilon^{\pm\alpha_j}] = (\alpha_i^\vee, \pm\alpha_j)\epsilon^{\pm\alpha_j}.$$

This gives (4.2.4) for the case $k = 0$ as required.

Now we assume that $k \in \mathbf{Z}_+$. We have

$$[\alpha_i^\vee(k), X(\pm\alpha_j, z)]$$

$$= [\alpha_i^\vee(k), \epsilon^{\pm\alpha_j} z^{\pm\alpha_j} E^-(\pm\alpha_j, z) W_{\pm\alpha_j}(z) E^+(\pm\alpha_j, z)]$$

$$= \epsilon^{\pm\alpha_j} z^{\pm\alpha_j} [\alpha_i^\vee(k), E^-(\pm\alpha_j, z)] W_{\pm\alpha_j}(z) E^+(\pm\alpha_j, z)$$

$$+ \epsilon^{\pm\alpha_j} z^{\pm\alpha_j} E^-(\pm\alpha_j, z) W_{\pm\alpha_j}(z) [\alpha_i^\vee(k), E^+(\pm\alpha_j, z)].$$

By applying the formal rule (3.2.10), we have

$$[\alpha_i^\vee(k), E^-(\pm\alpha_j, z)]$$

$$= E^-(\pm\alpha_j, z)[\alpha_i^\vee(k), -\sum_{n\in\mathbf{Z}_-} \frac{\pm\alpha_j(n)}{n} z^{-n}]$$

$$= E^-(\pm\alpha_j, z) \sum_{n\in\mathbf{Z}_+} [\alpha_i^\vee(k), \pm\alpha_j(-n)] z^n/n$$

$$= E^-(\pm\alpha_j, z) \sum_{n\in\mathbf{Z}_+} k(\alpha_i^\vee, \pm\alpha_j)\delta_{k-n,0} \frac{c}{n} z^n$$

$$= E^-(\pm\alpha_j, z)(\alpha_i^\vee, \pm\alpha_j) z^k,$$

and

$$[\alpha_i^\vee(k), E^+(\pm\alpha_j, z)]$$

$$= E^+(\pm\alpha_j, z)[\alpha_i^\vee(k), -\sum_{n\in\mathbf{Z}_+} \frac{\pm\alpha_j(n)}{z} z^{-n}] = 0.$$

Therefore, we obtain

$$[\alpha_i^\vee(k), X(\pm\alpha_j, z)]$$

$$= (\alpha_i^\vee, \pm\alpha_j) z^k \epsilon^{\pm\alpha_j} z^{\pm\alpha_j} E^-(\pm\alpha_j, z) W_{\pm\alpha_j}(z) E^+(\pm\alpha_j, z) = \pm(\alpha_i^\vee, \alpha_j) z^k X(\pm\alpha_j, z).$$

as required.

Similarly, we can prove (4.2.4) for the case $k \in \mathbb{Z}_-$. This completes the proof of (4.2.4), which gives the relation (R3).

Now, we check the relation (R4) by proving (4.2.5). That is

$$[X(\alpha_i, z_1), X(-\alpha_j, z_2)]$$
$$= -\delta_{ij}\left(\alpha_i^\vee(z_2)\delta(\frac{z_2}{z_1}) + \frac{2}{(\alpha_i, \alpha_j)}(D\delta)(\frac{z_2}{z_1})c\right).$$

First, we consider the case for $i \neq j$. Note that $B_l^{(1)}$ has only one short simple root, thus, at least, one of the roots in $\{\alpha_i, \alpha_j\}$ is a long root. Therefore, (4.2.5) follows from Corollary 4.3.2 and Proposition 4.4.7.

Now, we assume that $i = j$. We divide the argument into twe cases by assuming $\alpha_i \in \Phi_L$, or $\alpha_i \in \Phi_S$.

If $\alpha_i \in \Phi_L$, then by Lemma 4.4.2, and noting that

$$P(z_1, z_2) = \epsilon(\alpha_i, -\alpha_i)z_1^{(\alpha_i, -\alpha_i)}(1 - \frac{z_2}{z_1})^{(\alpha_i, -\alpha_i)} - \epsilon(-\alpha_i, \alpha_i)z_2^{(\alpha_i, -\alpha_i)}(1 - \frac{z_1}{z_2})^{(\alpha_i, -\alpha_i)}$$
$$= -z_1^{-2}(1 - \frac{z_2}{z_1})^{-2} + z_2^{-2}(1 - \frac{z_1}{z_2})^{-2}$$
$$= -z_1^{-1}z_2^{-1}(D\delta)(\frac{z_2}{z_1}),$$

we obtain, from this and (4.4.7)

$$[X(\alpha_i, z_1), X(-\alpha_i, z_2)]$$
$$= -z_1^{\alpha_i}z_2^{-\alpha_i}E^-(\alpha_i, z_1)E^-(-\alpha_i, z_2)E^+(\alpha_i, z_1)E^+(-\alpha_i, z_2)(D\delta)(\frac{z_2}{z_1})$$
$$= -\left(\alpha_i(0) + \sum_{n\in\mathbb{Z}_-}\alpha_i(n)z_1^{-n} + \sum_{n\in\mathbb{Z}_+}\alpha_i(n)z_1^{-n}\right)\delta(\frac{z_2}{z_1}) - (D\delta)(\frac{z_2}{z_1})$$
$$= -\alpha_i(z_2)\delta(\frac{z_2}{z_1}) - (D\delta)(\frac{z_2}{z_1})$$
$$= -\left(\alpha_i^\vee(z_2)\delta(\frac{z_2}{z_1}) + \frac{2}{(\alpha_i, \alpha_i)}(D\delta)(\frac{z_2}{z_1})c\right),$$

as required, where we have used Lemma 1.5.1.

If $\alpha_i \in \Phi_S$, we note that

$$R(z_1, z_2) = \epsilon(\alpha_i, -\alpha_i) z_1^{(\alpha_i, -\alpha_i)} (1 - \frac{z_2}{z_1})^{(\alpha_i, -\alpha_i)} \sum_{m,n \in \mathbf{Z}} \omega_{2m+1} \omega_{2n+1} z_1^{-m} z_2^{-n}$$

$$-\epsilon(-\alpha_i, \alpha_i) z_2^{(\alpha_i, -\alpha_i)} (1 - \frac{z_1}{z_2})^{(\alpha_i, -\alpha_i)} \sum_{m,n \in \mathbf{Z}} \omega_{2n+1} \omega_{2m+1} z_1^{-m} z_2^{-n}$$

$$= z_1^{-1} (1 - \frac{z_2}{z_1})^{-1} \sum_{m,n \in \mathbf{Z}} \omega_{2m+1} \omega_{2n+1} z_1^{-m} z_2^{-n} - z_2^{-1} (1 - \frac{z_1}{z_2})^{-1} \sum_{m,n \in \mathbf{Z}} \omega_{2n+1} \omega_{2m+1} z_1^{-m} z_2^{-n}$$

$$= z_1^{-1} \sum_{k \geq 0} (\frac{z_2}{z_1})^k \sum_{m,n \in \mathbf{Z}} \omega_{2m+1} \omega_{2n+1} z_1^{-m} z_2^{-n} - z_2^{-1} \sum_{k \geq 0} (\frac{z_1}{z_2})^k \sum_{m,n \in \mathbf{Z}} \omega_{2n+1} \omega_{2m+1} z_1^{-m} z_2^{-n}$$

$$= \sum_{\substack{m,n \in \mathbf{Z} \\ k \geq 0}} \omega_{2m+1} \omega_{2n+1} z_1^{-m-k-1} z_2^{-n+k} - \sum_{\substack{m,n \in \mathbf{Z} \\ k \geq 0}} \omega_{2n+1} \omega_{2m+1} z_1^{-m+k} z_2^{-n-k-1}$$

$$= \sum_{i,j \in \mathbf{Z}} a_{ij} z_1^{-i} z_2^{-j},$$

where

$$a_{ij} = \sum_{k \geq 0} (\omega_{2i-2k-1} \omega_{2j+2k+1} - \omega_{2j-2k-1} \omega_{2i+2k+1})$$

$$= \begin{cases} \omega_{2i-1} \omega_{2j+1} + \omega_{2i-3} \omega_{2j+3} + \cdots + \omega_{2j+1} \omega_{2i-1}, & \text{if } i > j, \\ 0, & \text{if } i = j, \\ -(\omega_{2j-1} \omega_{2i+1} + \omega_{2j-3} \omega_{2i+3} + \cdots + \omega_{2i+1} \omega_{2j-1}, & \text{if } i < j. \end{cases}$$

But in all cases, we have

$$a_{ij} = -2i \delta_{i+j,0}.$$

Therefore, we obtain

$$R(z_1, z_2) = \sum_{i,j \in \mathbf{Z}} (-2i \delta_{i+j,0}) z_1^{-i} z_2^{-j} = -2 \sum_{i \in \mathbf{Z}} i (\frac{z_2}{z_1})^i$$

$$= -2(D\delta)(\frac{z_2}{z_1}),$$

and, by Lemma 4.4.4

$$[X(\alpha_i, z_1), X(-\alpha_i, z_2)]$$

$$= -2 z_1^{\alpha_i} z_2^{-\alpha_i} E^-(\alpha_i, z_1) E^-(-\alpha_i, z_2) E^+(\alpha_i, z_1) E^+(-\alpha_i, z_2) (D\delta)(\frac{z_2}{z_1})$$

$$= -2 \left(\alpha_i(0) + \sum_{n \in \mathbf{Z}_-} \alpha_i(n) z_1^{-n} + \sum_{n \in \mathbf{Z}_+} \alpha_i(n) z_1^{-n} \right) \delta(\frac{z_2}{z_1}) - 2(D\delta)(\frac{z_2}{z_1})$$

$$= -2\alpha_i(z_2)\delta(\frac{z_2}{z_1}) - 2(D\delta)(\frac{z_2}{z_1})$$

$$= -\left(\alpha_i^\vee(z_2)\delta(\frac{z_2}{z_1}) + \frac{2}{(\alpha_i,\alpha_i)}(D\delta)(\frac{z_2}{z_1})c\right),$$

as required, where we have used Lemma 1.5.1. This completes the proof of the relation (R4).

Finally, we prove the relation (R5) by checking the identity (4.2.6). First, if $\alpha_i \in \Phi_L$, then it follows from Proposition 4.4.7 that

$$[X(\alpha_i, z_1), X(\alpha_i, z_2)] = 0.$$

If $\alpha_i \in \Phi_S$, then the above identity follows from Lemma 4.4.4 and the following fact

$$R(z_1, z_2) = \epsilon(\alpha_i, \alpha_i) z_1^{(\alpha_i,\alpha_i)}(1 - \frac{z_2}{z_1})^{(\alpha_i,\alpha_i)} \sum_{m,n\in\mathbf{Z}} \omega_{2m+1}\omega_{2n+1} z_1^{-m} z_2^{-n}$$

$$-\epsilon(\alpha_i, \alpha_i) z_2^{(\alpha_i,\alpha_i)}(1 - \frac{z_1}{z_2})^{(\alpha_i,\alpha_i)} \sum_{m,n\in\mathbf{Z}} \omega_{2n+1}\omega_{2m+1} z_1^{-m} z_2^{-n}$$

$$= \epsilon(\alpha_i, \alpha_i)(z_1 - z_2) \sum_{m,n\in\mathbf{Z}} (\omega_{2m+1}\omega_{2n+1} + \omega_{2n+1}\omega_{2m+1}) z_1^{-m} z_2^{-n}$$

$$= \epsilon(\alpha_i, \alpha_i)(z_1 - z_2) \sum_{m,n\in\mathbf{Z}} (-2\delta_{2(m+n+1),0}) z_1^{-m} z_2^{-n}$$

$$= -2\epsilon(\alpha_i, \alpha_i) z_2(z_1 - z_2) \sum_{m\in\mathbf{Z}} (\frac{z_2}{z_1})^m$$

$$= -2\epsilon(\alpha_i, \alpha_i) z_2(z_1 - z_2)\delta(\frac{z_2}{z_1}) = 0.$$

Likewise, one can prove the identity $[X(-\alpha_i, z_1), X(-\alpha_i, z_2)] = 0$ by a similar argument as above. Thus we have completed the proof of the first identity in (R5).

Now, we check the last two identities in (4.2.6), that is

$$\left.\begin{array}{r}[X(\alpha_i, z_p), \cdots, X(\alpha_i, z_2), X(\alpha_j, z_1)] = 0 \\ [X(-\alpha_i, z_p), \cdots, X(-\alpha_i, z_2), X(-\alpha_j, z_1)] = 0\end{array}\right\} \quad i \neq j, \tag{4.4.11}$$

where $p = -A_{ji} + 2$.

Since $i \neq j$, and $B_l^{(1)}$ has only one short simple root, then, one of the roots in $\{\alpha_i, \alpha_j\}$ is long. We divide the proof of (4.4.11) into two cases.

Case 1. If $\alpha_i \in \Phi_L$, $\alpha_j \in \Phi$ and $i \neq j$.

In this case, we have $(\alpha_i, \alpha_j) = 0$ or -1. But, if $(\alpha_i, \alpha_j) = 0$, then $p = -A_{ji}+2 = 2$, and (4.4.11) follows from Corollary 4.3.2 and Proposition 4.4.7. If $(\alpha_i, \alpha_j) = -1$, then $p = -A_{ji} + 2 = -\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} + 2 = 3$. Thus, by Corollary 4.3.2 and Proposition 4.4.5, we have

$$
\begin{aligned}
&[X(\pm\alpha_i, z_3), [X(\pm\alpha_i, z_2), X(\pm\alpha_j, z_1)]] \\
&= [X(\pm\alpha_i, z_3), \epsilon(\pm\alpha_i, \pm\alpha_j)X(\pm(\alpha_i + \alpha_j), z_1)\delta(\frac{z_1}{z_2})] \\
&= \epsilon(\pm\alpha_i, \pm\alpha_j)[X(\pm\alpha_i, z_3), X(\pm(\alpha_i + \alpha_j), z_1)]\delta(\frac{z_1}{z_2}),
\end{aligned}
$$

where, by using of Proposition 4.4.7

$$[X(\pm\alpha_i, z_3), X(\pm(\alpha_i + \alpha_j), z_1)] = 0,$$

as $\alpha_i + \alpha_j \in \Phi_L$, and $(\pm\alpha_i, \pm(\alpha_i + \alpha_j)) = 2 - 1 > 0$. This completes the proof of case 1.

Case 2 If $\alpha_i \in \Phi_S$, $\alpha_j \in \Phi_L$.

In this case, we have $(\alpha_i, \alpha_j) = 0$, or -1. As in Case 1, if $(\alpha_i, \alpha_j) = 0$, then $p = 2$ and (4.4.11) follows from Proposition 4.4.7. If $(\alpha_i, \alpha_j) = -1$, then $p = 4$. Thus, by Proposition 4.4.5, we have

$$
\begin{aligned}
&[X(\pm\alpha_i, z_4), X(\pm\alpha_i, z_3), X(\pm\alpha_i, z_2), X(\pm\alpha_j, z_1)] \\
&= [X(\pm\alpha_i, z_4), X(\pm\alpha_i, z_3), -\epsilon(\pm\alpha_i, \pm\alpha_j)X(\pm(\alpha_i + \alpha_j), z_1)\delta(\frac{z_1}{z_2})] \\
&= -\epsilon(\pm\alpha_i, \pm\alpha_j)[X(\pm\alpha_i, z_4), X(\pm\alpha_i, z_3), X(\pm(\alpha_i + \alpha_j), z_1)]\delta(\frac{z_1}{z_2}) \\
&= -\epsilon(\pm\alpha_i, \pm\alpha_j)[X(\pm\alpha_i, z_4), -2\epsilon(\pm\alpha_i, \pm(\alpha_i + \alpha_j))X(\pm(2\alpha_i + \alpha_j), z_1)\delta(\frac{z_1}{z_3})]\delta(\frac{z_1}{z_2}) \\
&= 2\epsilon(\pm\alpha_i, \pm\alpha_j)\epsilon(\pm\alpha_i, \pm(\alpha_i + \alpha_j))[X(\pm\alpha_i, z_4), X(\pm(2\alpha_i + \alpha_j), z_1)]\delta(\frac{z_1}{z_3})\delta(\frac{z_1}{z_2}),
\end{aligned}
$$

which equals zero by using of Proposition 4.4.7, since $2\alpha_i + \alpha_j \in \Phi_L$, and $(\alpha_i, 2\alpha_i + \alpha_j) \geq 0$.

Therefore, we have completed the proof of Theorem 4.3.6

§4.5 Vertex Operator Representation of the Clifford Algebra $\mathcal{W}$

Recall that $Q = \oplus_{i=0}^{l+1}\mathbb{Z}\alpha_i = Q(B_l^{(1)}) \oplus (\mathbb{Z}\alpha_{l+1})$, also note that

$$e^{\alpha_i}e^{\alpha_{l+1}} = e^{\alpha_{l+1}}e^{\alpha_i},$$

$$\epsilon(\alpha_i, \alpha_{l+1}) = \epsilon(\alpha_{l+1}, \alpha_i),$$

for $i = 0, 1, \dots, l$, and we have

$$\mathbb{C}[Q] = \mathbb{C}[Q(B_l^{(1)})] \otimes \mathbb{C}[\mathbb{Z}\alpha_{l+1}]. \tag{4.5.1}$$

Moreover, it is clear that

$$H_0 = \left(\oplus_{i=0}^{l+1}\mathbb{C}\beta_i\right) \oplus \mathbb{C}d = \left(\oplus_{i=0}^{l}\mathbb{C}\alpha_i\right) \oplus \mathbb{C}d \oplus \mathbb{C}\alpha_{l+1},$$

thus, this gives

$$\mathcal{S}(\hat{\mathcal{H}}_0^-) \simeq \mathcal{S}(\hat{\mathcal{H}}^-) \otimes \mathcal{S}(\mathbb{C}\alpha_{l+1}^-), \tag{4.5.2}$$

where $\mathbb{C}\alpha_{l+1}^- = \oplus_{n\in\mathbb{Z}_-}\mathbb{C}\alpha_{l+1}(n)$, and

$$\hat{\mathcal{H}} = \left(\oplus_{n\in\mathbb{Z}\setminus\{0\}}H(n)\right) \oplus \mathbb{C}c,$$

here $H(n)$ is a linear copy of $H = \left(\oplus_{i=1}^{l}\mathbb{C}\alpha_i\right) \oplus \mathbb{C}d$, for $n \in \mathbb{Z}$.

From (4.5.1) and (4.5.2), we have

$$\begin{aligned} V &= \mathbb{C}[Q] \otimes \mathcal{S}(\hat{\mathcal{H}}_0^-) \\ &\simeq \{\mathbb{C}[Q(B_l^{(1)})] \otimes \mathbb{C}[\mathbb{Z}\alpha_{l+1}]\} \otimes \{\mathcal{S}(\hat{\mathcal{H}}^-) \otimes \mathcal{S}(\mathbb{C}\alpha_{l+1}^-)\} \\ &\simeq \mathbb{C}[Q(B_l^{(1)})] \otimes \mathcal{S}(\hat{\mathcal{H}}^-)] \otimes \{\mathbb{C}[\mathbb{Z}\alpha_{l+1}] \otimes \mathcal{S}(\mathbb{C}\alpha_{l+1}^-)\}. \end{aligned}$$

Therefore, we obtain

Lemma 4.5.1 The operators e^{α_i}, z^{α_i}, for $i = 0, 1, \dots, l$, act only on $\mathbb{C}[Q(B_l^{(1)})]$, the operators ω_{2j+1}, for $j \in \mathbb{Z}$, act only on $\{\mathbb{C}[\mathbb{Z}\alpha_{l+1}] \otimes \mathcal{S}(\mathbb{C}\alpha_{l+1}^-)\}$, and $E^{\pm}(\alpha, z)$, for $\alpha \in \Phi(B_l^{(1)})$, act only on the symmtric algebra $\mathcal{S}(\hat{\mathcal{H}}^-)$.

$\square$

Recall that

$$(4.5.3) \qquad W(z) = z^{\frac{1}{2}}(Y(\alpha_{l+1}, z) - Y(-\alpha_{l+1}, z)),$$

and the operators ω_j, ($j \in 2\mathbf{Z} + 1$), are defined by

$$W(z) := \sum_{j \in \mathbf{Z}} \omega_{2j+1} z^{-j}.$$

Thus, from Lemma 4.5.1 and Lemma 4.4.1, we have

Theorem 4.5.2 The vector space $\mathbf{C}[\mathbf{Z}\alpha_{l+1}] \otimes S(\mathbf{C}\alpha_{l+1}^{-})$ affords a representation of the Clifford algebra $\mathcal{W}$, which is spanned by the elements ω_j, ($j \in 2\mathbf{Z} + 1$), and such that $\omega_i\omega_j + \omega_j\omega_i = -2\delta_{i+j,0}$ for $i, j \in 2\mathbf{Z} + 1$, by the vertex operator $W(z)$ defined by (4.5.3).

$\square$

§4.6 Second Construction of the Toroidal Lie Algebra of Type B_l

The first construction of the toroidal Lie algebra of type B_l given in Section 4.3, suggests a direct construction of this Lie algebra by immediately introducing the Clifford algebra $\mathcal{W}$ and its standard irreducible module into the picture. This construction indeed generalizes the Lepowsky-Primc construction [LP1] of the level one standard module of $B_l^{(1)}$ to the toroidal case.

Let $\delta = \sum_{i=0}^{l} n_i\alpha_i$ be the null root with $n_0 = 1$ of the affine Lie algebra $B_l^{(1)}$. We assume that the notation is chosen so that $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$ are long roots, α_l is short in $\Phi(B_l^{(1)})$.

Let $\Gamma = Q(B_l^{(1)}) \oplus \mathbf{Z}d$, where d is a symbol. We extend the symmetric form $(\cdot, \cdot)$ to Γ bilinearly by defining

$$(4.6.1) \qquad (d, \delta) = 1, \qquad (d, d) = 0 = (d, \alpha_i),$$

for $i = 1, 2, \ldots, l$, and extending bilinearly.

Let $H = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$, and extend the form $(\cdot,\cdot)$ to H so that it is $\mathbb{C}$-bilinearly. Thus the form $(\cdot,\cdot)$ defines a nondegenerate symmetric bilinear form on the vector space H. Let $H(n)$ be the linear copy of H by $a(n) \mapsto a$ for $a \in H$, $n \in \mathbb{Z}$.

We define a vector space

$$(4.6.2) \qquad \check{\mathcal{H}} = (\oplus_{k\in\mathbb{Z}} H(n)) \oplus \mathbb{C}c,$$

where c is a symbol. We equip $\check{\mathcal{H}}$ with an anticommutitive multilication $[\cdot,\cdot] : \check{\mathcal{H}} \times \check{\mathcal{H}} \to \check{\mathcal{H}}$ by defining

$$(4.6.3) \qquad [\alpha(k), \beta(m)] = k(\alpha, \beta)\delta_{k+m,o}c,$$

$$[c, \check{\mathcal{H}}] = 0,$$

for $\alpha, \beta \in H$, $k, m \in \mathbb{Z}$. Then $\check{\mathcal{H}}$ forms a Lie algebra which contains a Heisenberg subalgebra

$$(4.6.4) \qquad \hat{\mathcal{H}} = \left(\oplus_{k\in\mathbb{Z}\setminus\{0\}} H(n)\right) \oplus \mathbb{C}c.$$

Let $\mathcal{S}(\hat{\mathcal{H}}^-)$ be the symmetric algebra on $\hat{\mathcal{H}} := \oplus_{k\in\mathbb{Z}_-} H(n)$. Then $\hat{\mathcal{H}}$ has a standard irreducible representation on $\mathcal{S}(\hat{\mathcal{H}}_-)$ by requiring c acts as 1, $a(-m)$ acts as mutiplication operator, and $a(m)$ acts as differential operator for which

$$(4.6.5) \qquad a(m).b(-n) = m(a, b)\delta_{m-n,0},$$

for $m, n \in \mathbb{Z}_+$, $a, b \in H$.

Let $\epsilon : Q(B_l^{(1)}) \times Q(B_l^{(1)}) \to \{\pm 1\}$ be the map defined in Corollary 4.3.2, that is

$$(4.6.6) \qquad \epsilon(\alpha_i, \alpha_j) = \begin{cases} 1, & \text{if } i < j, \\ (-1)^{(\alpha_i,\alpha_i)-1}, & \text{if } i = j, \\ (-1)^{(\alpha_i,\alpha_j)}, & \text{if } i > j, \end{cases}$$

and

$$\epsilon(\sum_i m_i\alpha_i, \sum_i n_i\alpha_i) = \prod_{0\le i,j\le l} (\epsilon(\alpha_i, \alpha_j))^{m_i n_j}.$$

for $\sum_i m_i\alpha_i, \sum_i n_i\alpha_i \in Q(B_l^{(1)})$. Thus ϵ satisfies the 2-cocycle condition

$$\epsilon(\alpha,\beta)\epsilon(\alpha+\beta,\gamma) = \epsilon(\alpha,\beta+\gamma)\epsilon(\beta,\gamma), \tag{4.6.7}$$

for $\alpha,\beta,\gamma \in Q(B_l^{(1)})$. In particular, we have

$$\epsilon(\alpha_i,\pm\alpha_j)\epsilon(\pm\alpha_j,\alpha_i) = \begin{cases} 1, & \text{if } i = j, \\ (-1)^{(\alpha_i,\alpha_j)}, & \text{if } i \neq j. \end{cases} \tag{4.6.8}$$

Using ϵ, we define a twisted group algebra $\mathbb{C}[Q(B_l^{(1)})]$ of the additive group $Q(B_l^{(1)})$, with base elements of the form e^α for $\alpha \in Q(B_l^{(1)})$, and the multiplication

$$e^\alpha e^\beta = \epsilon(\alpha,\beta)e^{\alpha+\beta}, \tag{4.6.9}$$

for $\alpha,\beta \in Q(B_l^{(1)})$.

Let $\mathcal{W}$ be the Clifford algebra with generators ω_i for $i \in 2\mathbf{Z}+1$, and the relation

$$\omega_i\omega_j + \omega_j\omega_i = -2\delta_{i+j,0}, \tag{4.6.10}$$

for $i,j \in 2\mathbf{Z}+1$.

As in Chapter 1, we let $\Lambda(\mathcal{W}^-)$ be the exterior algebra on the generators ω_i for $i \in 2\mathbf{Z}_- + 1$. Then $\Lambda(\mathcal{W}^-)$ gives the standard irreduciable module of the Clifford algebra $\mathcal{W}$, with the action defined by

$$\omega_j.1 = 0, \qquad \omega_{-j}.\omega = \omega_{-j} \wedge \omega,$$

for $j \in 2\mathbf{N}+1$, and $\omega \in \Lambda(\mathcal{W}^-)$. These also imply

$$\omega_i(\omega_{-j} \wedge \omega) = -2\delta_{i-j,0}\omega - \omega_{-j} \wedge (\omega_i.\omega),$$

for $i,j \in 2\mathbf{N}+1$.

As in Chapter 1 and Chapter 3, we put

$$\omega_i = \delta_{i,0}, \qquad \text{for } i \in 2\mathbf{Z}. \tag{4.6.11}$$

Now we form the full Fock space

$$M = \mathbb{C}[Q(B_l^{(1)})] \otimes S(\mathcal{H}^-) \otimes \Lambda(\mathcal{W}^-). \tag{4.6.12}$$

We define the actions of $\tilde{\mathcal{H}}$. $\mathbf{C}[Q(B_l^{(1)})]$ and $\mathcal{W}$ on M by

$$(4.6.13) \qquad a(m).e^\gamma \otimes u \otimes \omega = e^\gamma \otimes a(m).u \otimes \omega.$$

$$a(0).e^\gamma \otimes u \otimes \omega = (a.\gamma)e^\gamma \otimes u \otimes \omega.$$

$$e^\alpha.e^\gamma \otimes u \otimes \omega = \epsilon(\alpha.\gamma)e^{\alpha+\gamma} \otimes u \otimes \omega.$$

$$\mu.e^\gamma \otimes u \otimes \omega = e^\gamma \otimes u \otimes \mu.\omega.$$

for $a \in H. \alpha \in Q(B_l^{(1)}). \mu \in \mathcal{W}. m \in \mathbf{Z} \setminus \{0\}$ and $e^\gamma \otimes u \otimes \omega \in M$.

To get a representation of the toroidal Lie algebra $\mathcal{L}(A)$ of type B_l by vertex operators on the Fock space M. we define the vertex operators as follows

$$(4.6.14) \qquad X(\alpha. z) = e^\alpha z^\alpha E^-(\alpha. z)W_\alpha(z)E^+(\alpha. z).$$

for $\alpha \in Q(B_l^{(1)})$. where

$$(4.6.15) \qquad E^\pm(\alpha. z) = \exp(-\sum_{n\in\mathbf{Z}_\pm} \frac{\alpha(n)}{n} z^{-n}).$$

$$W_\alpha(z) = \sum_{m\in\mathbf{Z}} \omega_{2m+(\alpha.\alpha)} z^{-m}.$$

and the operator z^α is defined by

$$(4.6.16) \qquad z^\gamma.e^\gamma \otimes u \otimes \omega = z^{(\alpha.\gamma)}e^\gamma \otimes u \otimes \omega.$$

for $e^\gamma \otimes u \otimes \omega \in M$.

It is clear that. if $(\alpha.\alpha) \in 2\mathbf{Z}$. then [cf. (4.6.11)]

$$W_\alpha(z) = z^{\frac{(\alpha.\alpha)}{2}}.$$

Therefore. if $(\alpha.\alpha) \in 2\mathbf{Z}$. the vertex operator (4.6.14) can be written as follows

$$(4.6.17) \qquad X(\alpha. z) = z^{\frac{(\alpha.\alpha)}{2}} e^\alpha z^\alpha E^-(\alpha. z)E^+(\alpha. z).$$

In particular. we have

$$X(\alpha_i. z) = z^{\frac{(\alpha_i.\alpha_i)}{2}} e^{\alpha_i} z^{\alpha_i} E^-(\alpha_i. z)E^+(\alpha_i. z).$$

for $i = 0, 1, \ldots, l-1$, and

$$X(\alpha_l, z) = e^{\alpha_l} z^{\alpha_l} E^-(\alpha_l, z) W(z) E^+(\alpha_l, z),$$

where

$$W(z) = \sum_{j \in \mathbf{Z}} \omega_{2j+1} z^{-j}.$$

We may formally expand the vertex operator $X(\alpha, z)$ into formal power series of the form

$$X(\alpha, z) = \sum_{m \in \mathbf{Z}} x_m(\alpha) z^{-m}, \tag{4.3.8}$$

where the moment operators $x_m(\alpha)$, for $m \in \mathbf{Z}$, act on M.

Repeating the arguments as in Section 4.4, one can easily see the following result is true.

Theorem 4.6.1 The Lie algebra $t(A)$ of the operators on M generated by the moment operators $x_m(\alpha_i)$, $i = 0, 1, \ldots, l$, $m \in \mathbf{Z}$ [cf. (4.6.18)] gives a representation of the toroidal Lie algebra of type B_l (in two variables) for $l \geq 3$.

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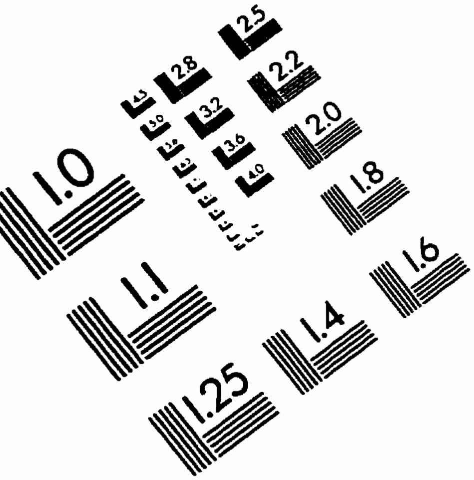
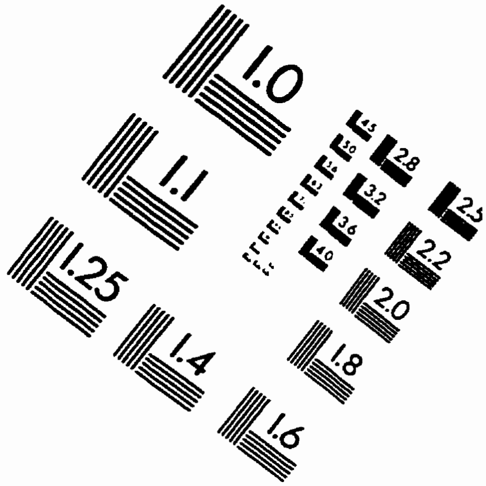
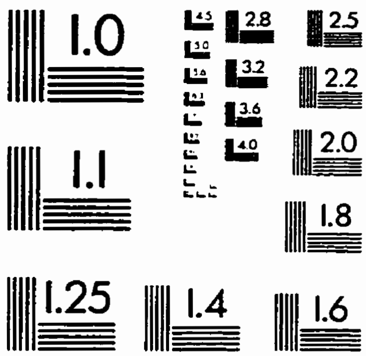
IMAGE EVALUATION TEST TARGET (QA-3)







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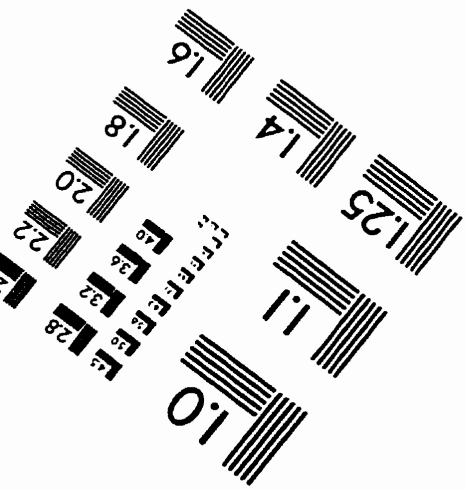

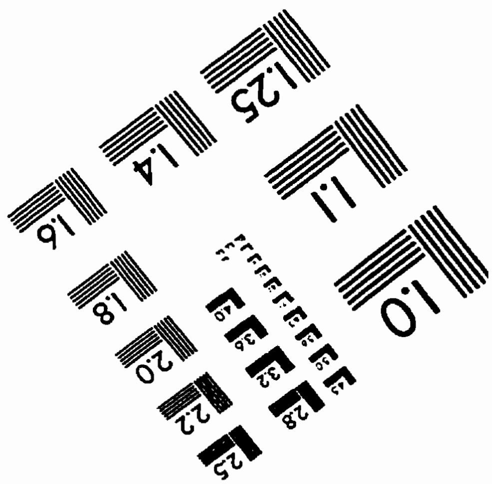

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